MATHEMATICS FOR ENGINEERS

PART II

The Directly-Useful **D.U.** Technical Series

Founded by the late WILFRID J. LINEHAM, B.Sc., M.Inst.C.E.
General Editor: JOHN L. BALE.

MATHEMATICS FOR ENGINEERS

PART II

including

DIFFERENTIAL AND INTEGRAL CALCULUS
SPHERICAL TRIGONOMETRY
AND MATHEMATICAL PROBABILITY

by

W. N. ROSE

B.Sc. Eng. (Lond.)

Late Head of Mathematics Department at the Borough Polytechnic Institute.

LONDON
CHAPMAN & HALL LTD
37 ESSEX STREET W.C.2

First Edition . . 1920
Second ,, . . 1923

CATALOGUE No. 167/4.

Made and printed by offset in Great Britain by
William Clowes and Sons, Limited, London and Beccles

FOUNDER'S NOTE

THE DIRECTLY-USEFUL TECHNICAL SERIES requires a few words by way of introduction. Technical books of the past have arranged themselves largely under two sections: the Theoretical and the Practical. Theoretical books have been written more for the training of college students than for the supply of information to men in practice, and have been greatly filled with problems of an academic character. Practical books have often sought the other extreme, omitting the scientific basis upon which all good practice is built, whether discernible or not. The present series is intended to occupy a midway position. The information, the problems and the exercises are to be of a directly-useful character, but must at the same time be wedded to that proper amount of scientific explanation which alone will satisfy the inquiring mind. We shall thus appeal to all technical people throughout the land, either students or those in actual practice.

CONTENTS

CHAPTER V

CHAPTER VI

CHAPTER VII

CHAPTER VIII

MATHEMATICS FOR ENGINEERS

PART II

INTRODUCTORY

THE subject-matter of this volume presents greater difficulty than that of Part I. Many of the processes described herein depend upon rules explained and proved in the former volume; and accordingly it is suggested that, before commencing to read this work, special attention should first be paid to Part I, pp. 452–460, 463–467, 469–472, and pp. 273–299; whilst a knowledge of the forms of the curves plotted in Chapter IX should certainly prove of great assistance.

The **abbreviations** detailed below will be adopted throughout.

$\rightarrow$	stands for	"approaches."
$=$	" "	"equals" or "is equal to."
$+$	" "	"plus."
$-$	" "	"minus."
$\times$	" "	"multiplied by."
$\div$	" "	"divided by."
$\therefore$	" "	"therefore."
$\pm$	" "	"plus or minus."
$>$	" "	"greater than."
$<$	" "	"less than."
$\odot$	" "	"circle."
$\odot$ce	" "	"circumference."
$\propto$	" "	"varies as."
∞	" "	"infinity."
$\angle$	" "	"angle."
$\triangle$	" "	"triangle" or "area of triangle."
$\underline{4\|}$ or $4!$	" "	"factorial four"; the value being that of the product 1.2.3.4 or 24.
${}^nP_2$	" "	"the number of permutations of n things taken two at a time."
${}^nC_2$	" "	"the number of combinations of n things taken two at a time."
n_3	" "	$n(n-1)(n-2)$.

η	stands for	"efficiency."
α	" "	"angle in degrees."
θ	" "	"angle in radians."
I.H.P.	" "	"indicated horse-power."
B.H.P.	" "	"brake horse-power."
m.p.h.	" "	"miles per hour."
r.p.m.	" "	"revolutions per minute."
r.p.s.	" "	"revolutions per second."
I.V.	" "	"independent variable."
F.°	" "	"degrees Fahrenheit."
C.°	" "	"degrees Centigrade."
E.M.F.	" "	"electro-motive force."
I	" "	"moment of inertia."
E	" "	"Young's modulus of elasticity."
S_n	" "	"the sum to n terms."
S_∞	" "	"the sum to infinity (of terms)."
Σ	" "	"sum of."
B.T.U.	" "	"Board of Trade unit."
B.Th.U.	" "	"British thermal unit."
τ	" "	"absolute temperature."
μ	" "	"coefficient of friction."
$\sin^{-1} x$	" "	"the angle whose sine is x."
e	" "	"the base of Napierian logarithms."
g	" "	"the acceleration due to the force of gravity."
cms.	" "	"centimetres."
grms.	" "	"grammes."
$\lim_{x \to a} y$	" "	"the limit to which y approaches as x approaches the value a."
C. of G.	" "	"centre of gravity."
C. of P.	" "	"centre of pressure."
k	" "	"swing radius," or "radius of gyration."
j	" "	"$\sqrt{-1}$."
M.V.	" "	"mean value."
R.M.S.	" "	"root mean square."
$f'(x)$	" "	"the first derivative of a function of x."
$f''(x)$	" "	"the second derivative of a function of x."
$\frac{dy}{dx}$	" "	"the differential coefficient of y with regard to x."
$\int y dx$	" "	"the integral of y with respect to x as the I.V."
δ	" "	"difference of."
∂	" "	"partial differential."
D	" "	"the operation $\frac{d}{dx}$."
C.P.	" "	"candle-power."
M.S.C.P.	" "	"mean spherical candle-power."
ρ	" "	"density."
A.M.	" "	"arithmetic mean."

CHAPTER I

INTRODUCTION TO DIFFERENTIATION

THE seventeenth century will ever be remarkable for the number of great mathematicians that it produced, and still more so for the magnitude of the research accomplished by them. In the early part of the century Napier and Briggs had introduced their systems of logarithms, whilst Wallis and others directed their thoughts to the quadrature of curves, which they effected in some instances by expansion into series, although the Binomial Theorem was then unknown to them. In 1665 Newton, in his search for the method of quadrature, evolved what he termed to be a system of "fluxions" or flowing quantities: if x and y, say, were flowing quantities, then he denoted the velocity by which each of these fluents increased by $\dot{x}$ and $\dot{y}$ respectively. By the use of these new forms he was enabled to determine expressions for the tangents of curves, and also for their radii of curvature. At about the same time Leibnitz of Leipsic, also concerned with the same problem, arrived at practically the same system, although he obtained his tangents by determining "differences of numbers." To Leibnitz is due the introduction of the term "differential," and also the differential notation, viz., dx and dy for the differentials of x and y: he also in his expression for the summation of a number of quantities first wrote the symbol $\int$ his first idea being to employ the word "omnia" or its abbreviation "omn." Thus, if summing a number of quantities like x, he first wrote "omnia x," which he contracted to "omn. x," and later he modified this form to $\int x$.

Great controversy raged for some time as to the claims of Newton and Leibnitz to be called the inventor of the system of the "Calculus," which is the generic term for a classified collection of rules; but it is now generally conceded that the discoveries were independent, and were in fact the natural culmination of the research and discoveries of many minds.

The Calculus was further developed by Euler, Bernoulli, Legendre and many others, but until a very recent date it remained merely "a classified collection of rules": its true meaning and the wide field of its application were for long obscured.

Nowadays, however, a knowledge of the Calculus is regarded, particularly by the engineer, as a vital part of his mental equipment: its rules have been so modified as to become no serious tax on the memory, and the true significance of the processes has been presented in so clear a light that the study of the Calculus presents few difficulties even to the ultra-practical engineer.

This revolution of thought has been brought about entirely through the efforts of men who, realising the vast potentialities of the Calculus, have reorganised the teaching of the subject: they have clothed it and made it a live thing.

The Calculus may be divided into two sections, viz., those treating of *differentiation* and *integration* respectively. Differentiation, as the name suggests, is that part of the subject which is concerned with *differences*, or more strictly with the comparison of differences of two quantities. Thus the process of differentiation resolves itself into a calculation of rates of change; but the manner in which the rate of change is determined depends on the form in which the problem is stated. Thus, if the given quantities are expressed by the co-ordinates of a curve, the change of the ordinate compared with the change in the abscissa for any particular value of the abscissa is measured by the slope of the curve at the point considered.

Differentiation is really nothing more nor less than **the determination of rates of change or of slopes of curves.**

The term "rate of change" does not necessarily imply a "time rate of change," *i. e.*, a rate of change with regard to time, such as the rate at which an electric current is changing per second, or the rate at which energy is being stored per minute; but the change in one quantity may be compared with the change in any other quantity. As an illustration of this fact we may discuss the following example—

The velocity of a moving body was measured at various distances from its starting point and the results were tabulated, thus—

s (distance in feet) . .	0	5	12
v (velocity in feet per sec.)	10	14	15

To find the values of the "space rate of change of velocity" for the separate space intervals.

Considering the displacement from 0 to 5 ft., the change in the velocity corresponding to this change of position is 14−10, *i. e.*, 4 ft. per sec.

Hence— $\dfrac{\text{change of velocity}}{\text{change of position}} = \dfrac{14-10}{5-0} = \dfrac{4}{5} = \cdot 8$

or, the change of velocity per one foot change of position = ·8 ft. per sec., and rate of change of velocity = ·8 ft. per sec. per foot.

Again, if s varies from 0 to 12, the change of $v = 15-10 = 5$; or, the rate of change of velocity (for this period) $= \frac{5}{12}$ ft. per sec. per foot.

Similarly, the rate of change of v, whilst s ranges from 5 to 12,

$$= \frac{15-14}{12-5} = \frac{1}{7} \text{ ft. per sec. per foot.}$$

The rates of change have thus been found by comparing differences. The phrase "change of" occurs frequently in this investigation, and to avoid continually writing it a symbol is adopted in its place. The letter thus introduced is δ (delta), the Greek form of d, the initial letter of the word "difference": it must be regarded on all occasions as an abbreviation, and hence no operation must be performed upon it that could not be performed if the phrase for which δ stands was written in full. In other words, the ordinary rules applying to algebraic quantities, such as multiplication, division, addition or subtraction, would be incorrectly used in conjunction with δ.

Thus, mv (the formula for momentum) means m multiplied by v, or a mass multiplied by a velocity, whilst δv represents "the change of v," or if v is the symbol for velocity, δv = change of velocity.

Again, δt = change of time or change of temperature, as the case may be. Using this notation our previous statements can be written in the shorter forms: thus—

(1) As s changes from 0 to 5 $\qquad \delta v = 14-10 = 4$

$\qquad \delta s = 5-0 = 5$

and $\qquad \dfrac{\delta v}{\delta s} = \dfrac{4}{5} = \cdot 8$

(2) As s changes from 0 to 12 $\qquad \delta v = 15-10 = 5$

$\qquad \delta s = 12-0 = 12$

and $\qquad \dfrac{\delta v}{\delta s} = \dfrac{5}{12} = \cdot 417$

(3) As s changes from 5 to 12

$$\delta v = 15-14 = 1$$
$$\delta s = 12-5 = 7$$

and $$\frac{\delta v}{\delta s} = \frac{1}{7} = \cdot 143$$

It must be noted that we do not cancel δ from the numerator and denominator of the fraction $\frac{\delta v}{\delta s}$.

The final result in (1), viz., $\frac{\delta v}{\delta s} = \cdot 8$, as s changes from 0 to 5, needs further qualification. From the information supplied we cannot say with truth that the change in the velocity for each foot from 0 to 5 ft. is ·8 ft. per sec.: all that we know with certainty is that, as s changes from 0 to 5 ft., the *average* rate of change of velocity over this space period is ·8 ft. per sec. Supposing the change of velocity to be continuous over the period considered, the value of $\frac{\delta v}{\delta s}$ already obtained would be the *actual* rate of change of velocity at some point or points in the period considered.

It is usual to tabulate the values of the original quantities and their changes, and unless anything is given to the contrary the *average* values of the rate of change are written in the middle of the respective periods.

The table is set out thus—

s	v	δs	δv	$\frac{\delta v}{\delta s}$
0	10	—	—	—
—	—	5	4	$\frac{4}{5} = \cdot 8$
5	14	—	—	—
—	—	7	1	$\frac{1}{7} = \cdot 143$
12	15	—	—	—

To distinguish in writing between *average* and *actual* rates of change the notation employed is slightly modified, d being used in place of δ; $\frac{dv}{dt}$ thus representing an *actual* rate of change of velocity, and $\frac{\delta v}{\delta t}$ representing an *average* rate of change of velocity. Once again it must be emphasised that d must be treated strictly in association with the v or t, as the case may be, and dt does not mean $d \times t$, nor does $\frac{dv}{dt}$ give $\frac{v}{t}$.

Another example can now be considered to demonstrate clearly the distinction between an average and an actual rate of change.

For a body falling freely under the influence of gravity the values of the distances covered to the ends of the 1st, 2nd and 3rd seconds of the motion are as in the table—

t (secs.) .	0	1	2	3
s (feet) .	0	16·1	64·4	144·9

Find the average velocities during the various intervals of time, and also the actual velocities at the ends of the 1st, 2nd and 3rd seconds respectively.

The average velocities are found in the manner described before, *i. e.*, by the comparison of differences of space and time, and the results are tabulated, thus—

t	s	δs	δt	$v = \frac{\delta s}{\delta t}$
0	0	—	—	—
—	—	16·1	1	16·1
1	16·1	—	—	—
—	—	48·3	1	48·3
2	64·4	—	—	—
—	—	80·5	1	80·5
3	144·9	—	—	—

The average velocities, viz., the values in the last column, are written in the lines between the values of the time to signify that they are the averages for the particular intervals. As also it is known that in this case the velocity is increased at a uniform rate, it is perfectly correct to state that the actual velocities at the ends of ·5, 1·5 and 2·5 seconds respectively are given by the average velocities over the three periods and are 16·1, 48·3 and 80·5 ft. per sec.

We have thus found the actual velocities at the half seconds, but not those at the ends of the 1st, 2nd and 3rd seconds. The determination of these velocities introduces a most important process, illustrating well the elements of differentiation, and in consequence the investigation is discussed in great detail.

The student of Dynamics knows that the law connecting space and time, in the case of a falling body, is $s = \frac{1}{2}gt^2 = 16{\cdot}1t^2$, and

a glance at the table of values of s and t confirms this law; thus, when $t=2$, $s=64\cdot4$, which $=16\cdot1\times2^2$ or $16\cdot1t^2$.

To find the actual velocity at the end of the first second we must calculate the average velocities over small intervals of time in the neighbourhood of 1 sec., and see to what figure these velocities approach as the interval of time is taken smaller and smaller.

Thus if— $t=1 \qquad s=16\cdot1\times1^2=16\cdot1$

$t=1\cdot1 \qquad s=16\cdot1\times1\cdot1^2=19\cdot481$

$\delta s=19\cdot481-16\cdot1=3\cdot381 \qquad \delta t=1\cdot1-1=\cdot1$

and $$(\text{average})\ v=\frac{\delta s}{\delta t}=\frac{3\cdot381}{\cdot1}=33\cdot81$$

i. e., the average velocity over the interval of time 1 to 1·1 sec. is 33·81 ft. per sec. This value must be somewhere near the velocity at the end of the first second, but it cannot be the absolute value, since even in the short interval of time, viz., ·1 sec., the velocity has been increased by a measurable amount. A better approximation will evidently be found if the time interval is narrowed to ·01 sec.

Then— $t=1 \qquad s=16\cdot1$

$t=1\cdot01 \qquad s=16\cdot1\times1\cdot01^2=16\cdot42361$

$\delta s=\cdot32361 \qquad \delta t=\cdot01$

$$(\text{average})\ v=\frac{\delta s}{\delta t}=\frac{\cdot32361}{\cdot01}=32\cdot361$$

A value still nearer to the true will be obtained if the time interval is made ·001 sec. only.

$t=1 \qquad s=16\cdot1$

$t=1\cdot001 \qquad s=16\cdot1\times1\cdot001^2=16\cdot1322161$

$\delta s=\cdot0322161 \qquad \delta t=\cdot001$

and $$(\text{average})\ v=\frac{\delta s}{\delta t}=\frac{\cdot0322161}{\cdot001}=32\cdot2161$$

By taking still smaller intervals of time, more and more nearly correct approximations would be found for the velocity; the values of v all tending to 32·2, and thus we are quite justified in saying that when $t=1$, $v=32\cdot2$ ft. per sec.

Or, using the language of p. 458 (*Mathematics for Engineers*, Pt. I), we state that the limiting value of v as t approaches 1 is 32·2; a result expressed in the shorter form

$$(\text{average})\ v\rightarrow32\cdot2 \text{ as } \delta t\rightarrow0 \quad \text{when } t=1$$

where the symbol $\rightarrow$ means "approaches"

but $(\text{average})\ v=\frac{\delta s}{\delta t}$, and thus $\frac{\delta s}{\delta t}\rightarrow32\cdot2$ as $\delta t\rightarrow0$ when $t=1$

Again, an actual velocity is an average velocity over an extremely small interval of time; or, in other words, an actual velocity is the limiting value of an average velocity, so that—

$$\text{(actual) } v = \lim_{\delta t \to 0} \text{(average) } v$$

$$i.e., \quad \frac{ds}{dt} = \lim_{\delta t \to 0} \frac{\delta s}{\delta t}$$

By similar reasoning it could be proved that the actual velocity at the end of the 2nd second was 64·4 ft. per sec., and at the end of the 3rd second the velocity was 96·6 ft. per sec.

This example may usefully be continued a step further, by calculating the values of the acceleration; this being now possible since the velocities are known.

Tabulating as before—

t	v	δv	δt	$a = \frac{\delta v}{\delta t}$
1	32·2	—	—	—
—	—	32·2	1	32·2
2	64·4	—	—	—
—	—	32·2	1	32·2
3	96·6	—	—	—

and we note that the average acceleration is constant and is thus the actual acceleration.

Our results may now be grouped together in one table, in which some new symbols are introduced, for the following reason. A velocity is the rate of change of displacement, and is found by "differentiating space with regard to time," and an acceleration is the rate of change of velocity, and hence it is a rate of change of the change of position, and so implies a double differentiation. Thus whilst $\frac{ds}{dt}$ is called the first *derivative* or *differential coefficient* of s with regard to t, $\frac{dv}{dt}$ is the first derivative of v with regard to t and the second derivative of s with regard to t.

Then $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right)$, this last form being usually written as $\frac{d^2s}{dt^2}$ (spoken as d two s, dt squared); and it denotes that the operation of differentiating has been performed twice upon s.

The complete table of the values of the velocity and the acceleration reads—

s	t	δs	δt	$v = \frac{\delta s}{\delta t}$	$\delta v = \delta\left(\frac{\delta s}{\delta t}\right)$	δt	$a = \frac{\delta v}{\delta t} = \frac{\delta^2 s}{\delta t^2}$
0	0	—	—	—	—	—	—
—	—	16·1	1	16·1	—	—	—
16·1	1	—	—	—	32·2	1	32·2
—	—	48·3	1	48·3	—	—	—
64·4	2	—	—	—	32·2	1	32·2
—	—	80·5	1	80·5	—	—	—
144·9	3	—	—	—	—	—	—

The next example refers to a similar case, but is treated from the graphical aspect.

Example 1.—Experiments made with the rolling of a ball down an inclined plane gave the following results—

t (secs.) .	0	1	2	3
s (cms.) .	0	20	80	180

Draw curves giving the space, velocity and acceleration respectively at any time during the period 0 to 3 secs.

By plotting the given values, s vertically and t horizontally, the "space-time" curve or "displacement" curve is obtained; the curve being a parabola (Fig. 1).

Select any two points P and Q on the curve, not too far apart, and draw the chord PQ, the vertical QN and the horizontal PN.

Then the slope of the chord PQ $= \frac{QN}{PN}$.

Now PN may be written as δt since it represents a small addition to the value of t at P: also QN $= \delta s$,

so that— slope of chord PQ $= \frac{\delta s}{\delta t}$

but $\frac{\delta s}{\delta t}$ = average velocity between the times OM and OR, hence the average velocity is measured by the slope of a chord. Now let Q approach P, then the chord PQ tends more and more to lie along the tangent at P, and by taking Q extremely close to P the chord PQ and the tangent at P are practically indistinguishable

the one from the other; whilst in the limit the two lines coincide. Then since the slope of the chord PQ gives the value of $\frac{\delta s}{\delta t}$, and the limiting value of $\frac{\delta s}{\delta t}$ is $\frac{ds}{dt}$, it follows that the slope of the tangent expresses $\frac{ds}{dt}$; but the slope of a curve at any point is measured

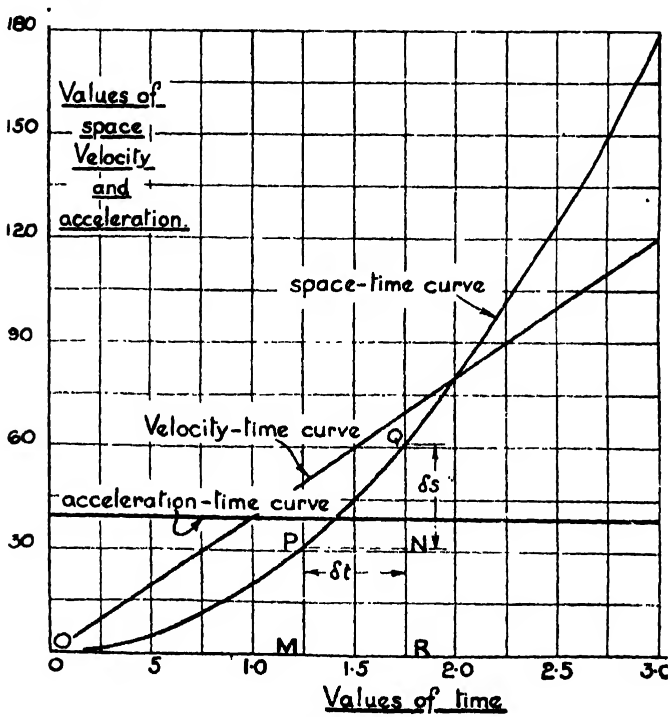


FIG. 1.

by the slope of its tangent at that point, and hence we have evolved the most important principle, viz., that differentiation is the determination of the slopes of curves.

[Incidentally it may be remarked that here is a good illustration of the work on limiting values; for the slope of a curve, or of the tangent to the curve, is the limiting value of the slope of the chord, *i. e.*, the value found when the extremities of the chord coincide; and this value does not take the indeterminate form $\frac{0}{0}$, as might at first sight be supposed, but is a definite figure.]

Thus the slope of the tangent at any point on the space-time curve measures the actual rate of change of the space with regard to time at that particular instant; or, in other words, the actual velocity at that instant. Hence by drawing tangents to the space-time curve at various points and calculating the slopes, a set of values of the velocity is obtained: these values are then plotted to a base of time and a new curve is drawn, which gives by its ordinate the value of the velocity at any time and is known as the "velocity-time" curve.

Since this curve is obtained by the calculation of slopes, or rates of change, it is designated a *derived* or *slope* curve; the original curve, viz., the space-time curve, being termed the *primitive.*

In the case under notice the velocity-time curve is a sloping straight line, and in consequence its slope is constant, having the value 40. Hence the derived curve, which is the acceleration-time curve, is a horizontal line, to which the ordinate is 40. There are thus the three curves, viz., the primitive or space-time curve, the *first derived* curve or the velocity-time curve, and the *second derived* curve or the acceleration-time curve.

Graphic Differentiation.—The accurate construction of slope curves is a most tedious business, for the process already described necessitates the drawing of a great number of tangents, the calculations of their respective slopes and the plotting of these values. There are, however, two modes of graphic differentiation, both of which give results very nearly correct provided that reasonable care is taken over their use.

Method 1 (see Fig. 2).—Divide the base into small elements, the lengths of the elements not being necessarily alike, but being so chosen that the parts of the curve joining the tops of the consecutive ordinates drawn through the points of section of the base are, as nearly as possible, straight lines. Thus, when the slope of the primitive is changing rapidly, the ordinates must be close together; and when the curve is straight for a good length, the ordinates may be placed well apart. Choose a pole P, to the left of some vertical OA, the distance OP being made a round number of units, according to the horizontal scale. Erect the mid-ordinates for all the strips.

Through P draw PA parallel to ab, the first portion of the curve, and draw the horizontal Ac to meet the mid-ordinate of the first strip in c. Then dc measures, to some scale, the slope of the chord ab, and therefore the slope of the tangent to the primitive

curve at m, or the average slope of the primitive from a to b, with reasonable accuracy.

Continue the process by drawing PM parallel to bl and Ms horizontal to meet the mid-ordinate of the second strip in s: then cs is a portion of the slope or derived curve.

Repeat the operations for all the strips and draw the smooth curve through the points c, s, etc.: then this curve is the curve of slopes.

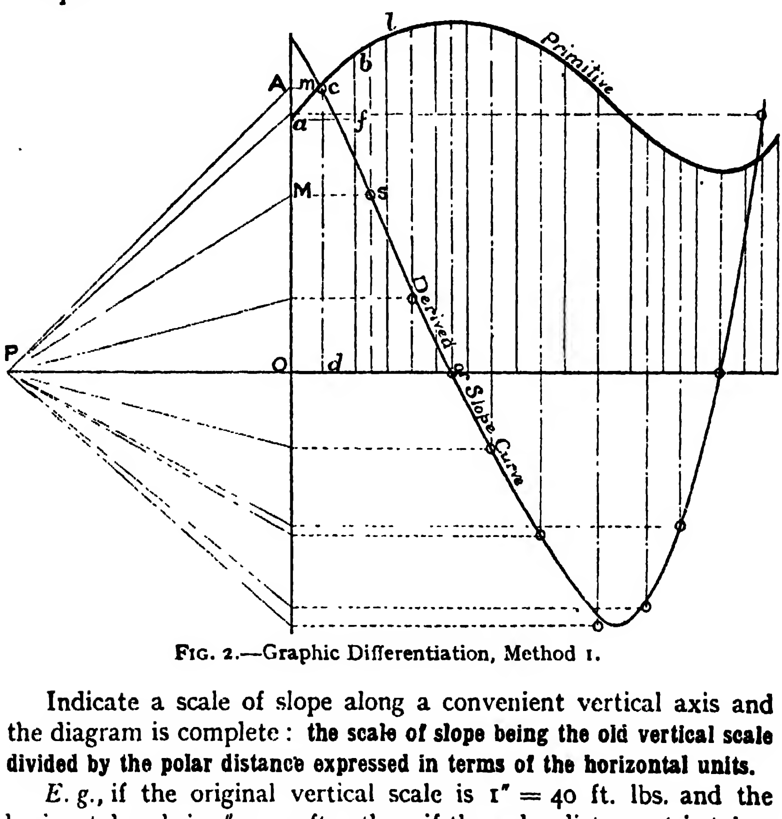


FIG. 2.—Graphic Differentiation, Method 1.

Indicate a scale of slope along a convenient vertical axis and the diagram is complete: **the scale of slope being the old vertical scale divided by the polar distance expressed in terms of the horizontal units.**

E. g., if the original vertical scale is $1'' = 40$ ft. lbs. and the horizontal scale is $1'' = 10$ ft.: then, if the polar distance p is taken as $2''$, *i. e.*, as 20 horizontal units,

the new vertical scale, or scale of slope, is $1'' = \dfrac{40 \text{ ft. lbs.}}{20 \text{ ft.}} = 2$ lbs.

Proof of the construction.—

The slope of the primitive curve at m = slope of curve ab

$$= \frac{bf}{af} = \frac{\text{OA}}{p} = \frac{cd}{p}$$

or, the ordinate dc, measured to the old scale, $= p \times$ the slope of the curve at m.

If, then, the original vertical scale is divided by p the ordinate dc, measured to the new scale, = slope of the curve at m.

The great disadvantage of this method is that parallels have to be drawn to very small lengths of line so that any slight error in the setting of the set square may quite easily be magnified in the drawing of the parallel. Hence, for accuracy, extreme care in draughtsmanship is necessary.

It should be observed that this method of graphic differentiation is the converse of the method of graphic integration described in

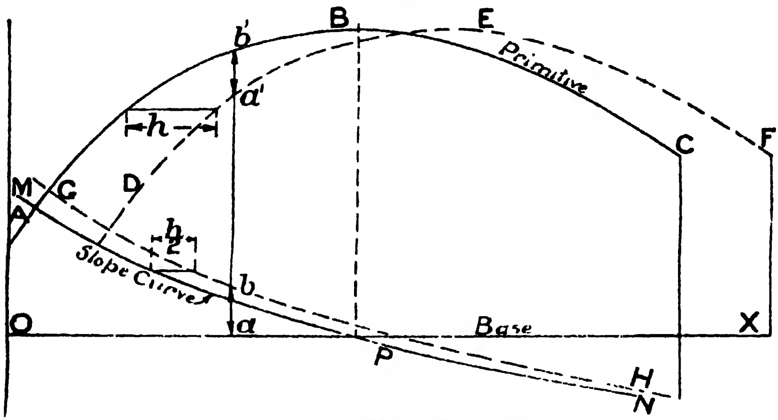


FIG. 3.—Graphic Differentiation, Method 2.

Chapter VII (Part I), and referred to in greater detail in Chapter V of the present volume.

Method 2.—Let ABC (Fig. 3) be the primitive curve.

Shift the curve ABC forward to the right a horizontal distance sufficiently large to give a well-defined difference between the curves DEF and ABC; but the horizontal distance, denoted by h, must not be great. From the straight line base OX set up ordinates which give the differences between the ordinates of the curves ABC and DEF, the latter curve being treated as the base: thus $ab = a'b'$. Join the tops of the ordinates so obtained to give the new curve GbH, and shift the curve GbH to the left a horizontal distance $= \frac{h}{2}$, this operation giving the curve MPN, which is the true slope or derived curve of the primitive ABC. Complete the diagram by adding a scale of slope, which is the old vertical scale divided by h (expressed in horizontal units).

This method can be still further simplified by the use of tracing paper, thus: Place the tracing paper over the diagram and trace the curve ABC upon it; move the tracing paper very carefully forward the requisite amount, viz., h, and with the dividers take the various differences between the curves, such as $a'b'$. Step off these differences from OX as base, but along ordinates $\frac{h}{2}$ units removed to the left of those on which the differences were actually measured: then draw the curve through the points and this is the slope curve.

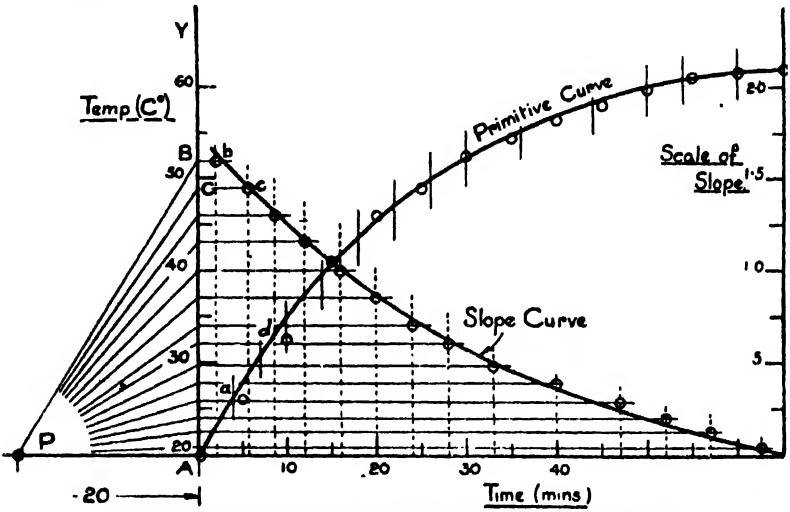


FIG. 3*a*.—Variation of Temperature of Motor Field Coils.

Examples on the use of these two methods now follow.

Example 2.—The temperature of the field coils of a motor was measured at various times during the passage of a strong current, with the following results—

Time (mins.)	0	5	10	15	20	25	30	35	40	45	50	55	60	65
Temperature (C.°)	20	26	32·5	41	46	49	52·5	54·5	56·5	58	59·5	61	61·7	62

Draw a curve to represent this variation of temperature, and a curve to show the rate at which the temperature is rising at any instant during the period of 65 mins.

The values of the temperature when plotted to a base of time give the primitive curve in Fig. 3*a*.

To draw the slope curve we first divide the base in such a way that the portions of the curve between consecutive ordinates have the same inclination for the whole of their lengths, *i. e.*, the elements of the curve are approximately straight lines. Thus, in the figure, there is no appreciable change of slope between A and *a*, or *a* and *d*. There is no need to draw the ordinates through the points of section for their full lengths, since the intersections with the primitive curve are all that is required. Next a pole P is chosen, 20 horizontal units to the left of A, and through P the line PB is drawn parallel to the portion of the curve A*a*. A horizontal B*b* cuts the mid-ordinate of the first strip at *b*, and *b* is a point on the slope or derived curve. The process is repeated for the second strip, PC being drawn parallel to *ad* and C*c* drawn horizontal to meet the mid-ordinate of the second strip in *c*, which is thus a second point on the slope curve. A smooth curve through points such as *b* and *c* is the slope curve, giving by its ordinates the rate of increase of the temperature; and it will be observed that the rate of increase is diminished until at the end of 65 mins. the rate of change of temperature is zero, thus indicating that at the end of 65 mins. the losses due to radiation just begin to balance the heating effect of the current.

Since the polar distance = 20 units, the scale of slope

$$= \frac{\text{original vertical scale}}{20};$$

and in the figure the original vertical scale is 1″ = 20 units; hence the scale of slope is 1″ = 1 unit; and this scale is indicated to the right of the diagram.

Example 3.—Plot the curve $y = x^2$, x ranging from 0 to 3, and use *Method* 2 to obtain the derived curve.

The values for the ordinates of the primitive curve $y = x^2$ are as in the table—

x	0	1	2	3
y	0	1	4	9

and the plotting of these gives the curve OAB in Fig. 4.

Choosing h as ·5 horizontal unit, the curve is first shifted forward this amount, and the curve CG results. The vertical differences between these two curves are measured, CG being regarded as the base curve, and are then set off from the axis of x as the base. Thus when $x = 3$, the ordinate of the curve OAB is 9 units, and that of CG is 6·25, so that the difference is 2·75, and this is the ordinate of the curve MN. By shifting the curve MN to the left by a distance $= \frac{\cdot 5}{2}$, *i. e.*, ·25 horizontal unit, the true slope curve ODE is obtained: this is a straight

line, as would be expected since the primitive curve is a "square" parabola.

As regards the scale of slope, the new vertical scale

$$= \frac{\text{old vertical scale,}}{h}$$

and since $h = \cdot 5$, the new vertical scale, or scale of slope, which is used when measuring ordinates of the curve ODE, is twice the original vertical scale.

The derived curve supplies much information about the primitive. Thus, when the ordinate of the derived curve is zero, *i. e.*, when the derived curve touches or cuts the horizontal axis, the

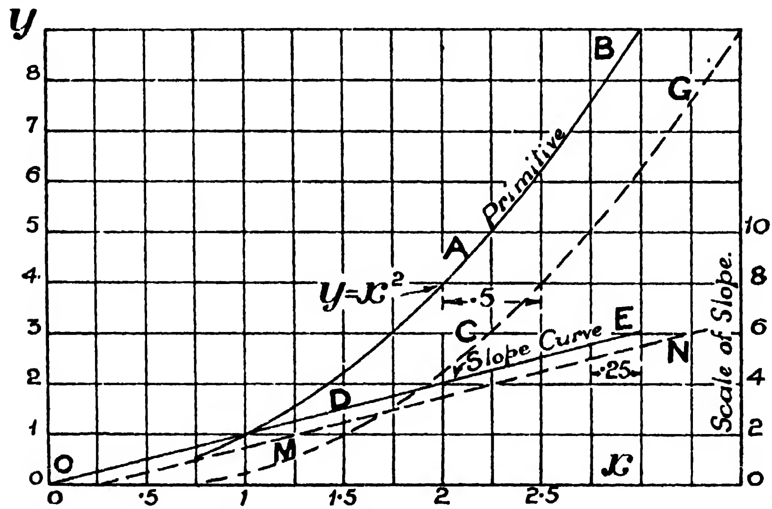


FIG. 4.—Graphic Differentiation.

slope of the primitive is zero; but if the slope is zero the curve must be horizontal, since it neither rises nor falls, and this is the case at a turning point, either maximum or minimum. Hence turning points on the primitive curve are at once indicated by zero ordinates of the slope curve.

Again, a positive ordinate of the derived curve implies a positive slope of the primitive, and thus indicates that in the neighbourhood considered the ordinate increases with increase of abscissa. Also a large ordinate of the slope curve indicates rapid change of ordinate of the primitive with regard to the abscissa.

This last fact suggests another and a more important one. By a careful examination of the primitive curve we see what is *actually*

happening, whilst the slope curve carries us further and tells us what is *likely to happen*. In fact, the rate at which a quantity is changing is very often of far greater importance than the actual value of the quantity; and as illustrations of this statement the following examples present the case clearly.

Example 4.—The following table gives the values of the displacement of a 21 knot battleship and the weight of the offensive and defensive factors, viz., armament, armour and protection. From these figures

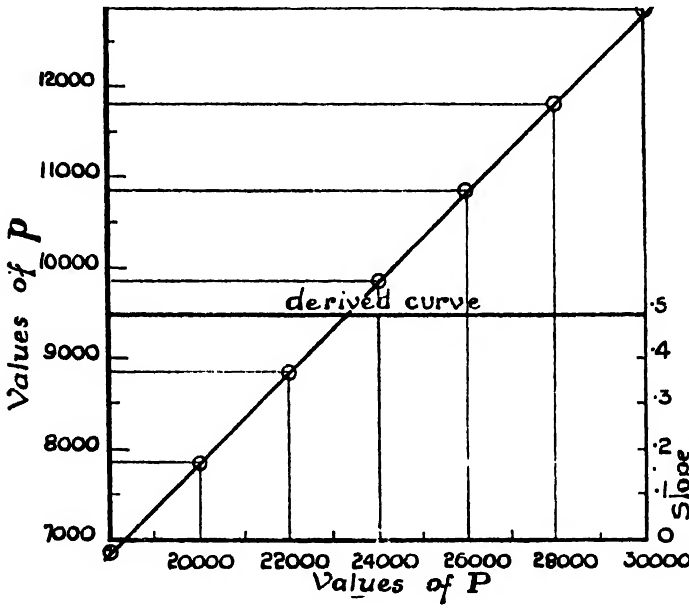


FIG. 5.—Displacement and Armament of Battleship.

calculate values of Q (ratio of armament, etc., to displacement) and q (rate of increase of armament, etc., with regard to displacement). Find also the values of $\frac{q}{Q}$.

Displacement P tons	18000	20000	22000	24000	26000	28000	30000
Armament, etc., p tons	6880	7850	8830	9820	10810	11820	12845

The values of Q are found by direct division and are—

P	18000	20000	22000	24000	26000	28000	30000
Q	·383	·392	·401	·409	·416	·422	·428

Values of q, *i. e.*, $\frac{dp}{dP}$, may be found by (a) construction of a slope curve, or (b) tabulating differences.

(a) *By construction of a slope curve.*—Plotting p along the vertical axis and P along the horizontal axis (see Fig. 5), we find that the points lie very nearly on a straight line. Hence the slope curve is a horizontal line, whose ordinate everywhere is the slope of the original line. By actual measurement the slope is found to be ·498 and thus the *average* value for $\frac{dp}{dP}$, over the range considered, is ·498. This average value of the rate of change does not, however, give as much information for our immediate purpose as the separate rates of change considered over the various small increases in the displacement.

(b) *By tabulation of differences,* as in previous examples—

P	p	δp	δP	$q = \frac{\delta p}{\delta P}$
18000	6880	—	—	—
—	—	970	2000	·485
20000	7850	—	—	—
—	—	980	2000	·490
22000	8830	—	—	—
—	—	990	2000	495
24000	9820	—	—	—
—	—	990	2000	·495
26000	10810	—	—	—
—	—	1010	2000	·505
28000	11820	—	—	—
—	—	1025	2000	·5125
30000	12845	—	—	—

Now $\frac{\delta p}{\delta P}$ = rate of increase of armament compared with displacement; as the displacement increases it is seen from the table of values that this ratio increases, and the questions then arise: "Does this increase coincide with an increase or a decrease in the values of Q, and if with one of these, what is the relation between the two changes?"

By tabulating the corresponding values of q and Q and calculating the values of $\frac{q}{Q}$, we obtain the following table (the values of Q at 19000, 21000, etc., being found from a separate plotting not shown here)—

P	19000	21000	23000	25000	27000	29000
p	·485	·490	·495	·495	·505	·5125
Q	·387	·396	·405	·412	·419	·426
$\frac{q}{Q}$	1·253	1·236	1·223	1·202	1·203	1·204

It will be seen by examination of this table that the fraction $\frac{q}{Q}$ decreases as ships are made larger: in other words, while the armament increases with the displacement, the increase is not so great as it should be for the size of the ship, since the weight of the necessary engines, etc., is greater in proportion to the weight of armament and protection for the larger than for the smaller ships.

Thus, other things being equal, beyond a certain point it is better to rely on a greater number of smaller ships than a few very large ones.

Example 5.—Friend gives the following figures as the results of tests on iron plates exposed to the action of air and water. The original plates weighed about 2·5 to 3 grms.

Plot these figures and obtain the rate curves for the two cases, these curves being a measure of the corrosion: comment on the results.

Time in days . .	2	7	13	19	26	32	37
In the light: loss of weight in grms. .	·0048	·031	·0645	·08	·093	·126	—
In the dark: loss of weight in grms. .	·0032	·0208	·037	·058	·0674	·0816	·0916

The two sets of values are plotted in Fig. 6, the respective curves being LLL for the plates exposed in the light, and DDD for those left in the dark. The effect of the action of light is very apparent from an examination of these curves. Next, the slope curves for the two cases are drawn, Method 2 being employed, but the intermediate steps are not shown. The curve *lll* is the slope curve for LLL, and *ddd* that for the curve DDD.

It will be observed that in both cases the rate of loss is great at the commencement, but decreases to a minimum value after 20 days exposure in the case of the curve *lll*, and after 25 days in the case of the curve *ddd*.

After these turning points have been reached the rate quickens,

the effect being very marked for the plates exposed to the light; and for these conditions the slope curve *lll* suggests that the corrosive action is a very serious matter, since it appears that the rate of loss must steadily increase.

A further extremely good illustration of the value of slope curves is found in connection with the cooling curves of metals. In the early days of the research in this branch of science, the cooling curve alone was plotted, viz., temperatures plotted to a base of

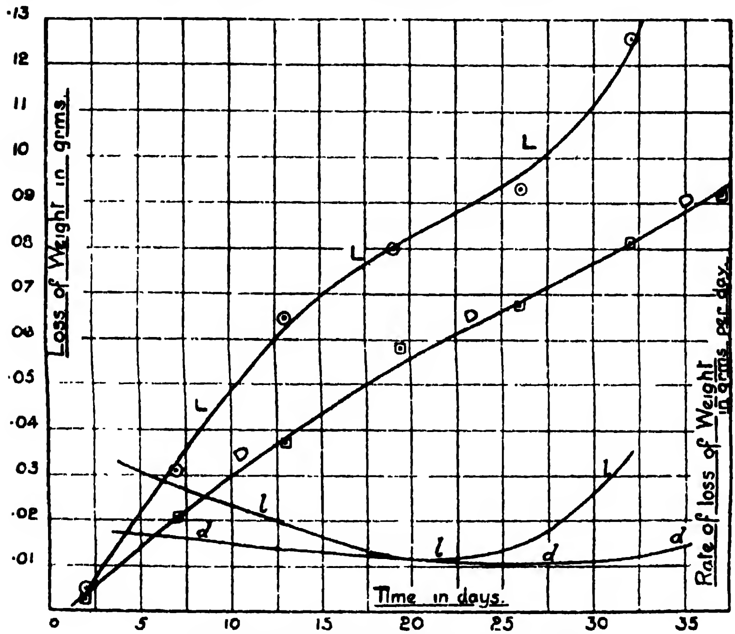


FIG. 6.—Tests on Corrosion of Iron Plate.

time. Later investigations, however, have shown that three other curves are necessary, viz., an inverse rate curve, a difference curve and a derived differential curve; the co-ordinates for the respective curves being—

(*a*) Temperature (θ)—time (t) curve; t horizontal and θ vertical.

(*b*) Inverse rate curve: $\frac{dt}{d\theta}$ horizontal and θ vertical. To obtain this curve from curve (*a*), the slopes must be very carefully calculated, and it must be remembered that these slopes are the measures of the inclinations to the vertical axis and not to the horizontal, *i. e.*, are values of $\frac{dt}{d\theta}$ and not $\frac{d\theta}{dt}$.

(*c*) θ vertical and $\theta-\theta_1$ horizontal: $\theta-\theta_1$ being the difference of temperature between the sample and a neutral body cooling under identical conditions.

(*d*) θ vertical and $\frac{d(\theta-\theta_1)}{d\theta}$ horizontal: this curve thus being the inverse rate curve of curve (*c*).

Exercises 1.—On Rates of Change and Derived Curves

1. What do the fractions $\frac{\delta s}{\delta t}$ and $\frac{ds}{dt}$ actually represent (s being a displacement, and t a time)? Take some figures to illustrate your answer.

2. Further explain the meanings of $\frac{\delta s}{\delta t}$ and $\frac{ds}{dt}$ by reference to a graph.

3. When an armature revolves in a magnetic field the E.M.F. produced depends on the rate at which the lines of force are being cut. Express this statement in a very brief form.

4. For a non-steady electric current the voltage V is equal to the resistance R multiplied by the current I, plus the self-inductance L multiplied by the time rate at which the current is changing. Express this in the form of an equation.

5. At a certain instant a body is 45·3 cms. distant from a fixed point. 2·14 seconds afterwards it is 21·7 cms. from this point. Find the average velocity during this movement. At what instant would your result probably measure the actual velocity?

6. At 3 ft. from one end of a beam the bending moment is 5 tons ft. At 3′ 2½″ from the same end it is 5·07 tons ft. If the shear is measured by the rate of change of bending moment, what is the average shearing force in this neighbourhood?

7. Tabulate the values of q, *i. e.*, $\frac{\delta p}{\delta P}$ for the following case, the figures referring to a battleship of 23 knots.

P	18000	20000	22000	24000	26000	28000	30000	32000
p	6170	7080	8000	8930	9890	10855	11820	12810

8. Tabulate the values of $\frac{q}{Q}$ for the case of a battleship of 25 knots from the following—

P	18000	20000	22000	24000	26000	28000	30000	32000
p	5210	6050	6910	7790	8660	9550	10460	11370

$$q = \frac{\delta p}{\delta P} \text{ and } Q = \frac{p}{P}.$$

9. Tabulate the values of the velocity and the acceleration for the following case—

Space (feet)	1	2·4	4·4	6	7·6	11·2	15·6	20·4
Time (secs.)	·2	·4	·6	·7	·8	1	1·2	1·4

10. Plot the space-time curve for the figures given in Question 9 and by graphic differentiation obtain the velocity-time and the acceleration-time curves.

11. Plot the curve $y = \cdot 5x^3$ from $x = -2$ to $x = +4$ and also its derived curve. What is the ordinate of the latter when $x = 1\cdot 94$?

12. Given the following figures for the mean temperatures of the year (the average for 50 years), draw a curve for the rate of change of temperature and determine at what seasons of the year it is most rapid in either direction.

Time (intervals of ½ month)	0	1	2	3	4	5	6	7	8	9	10	11
Temperature	38·6	37·9	38·4	39·8	38·5	39·5	40·3	40·7	41·5	45·5	45·5	48·5

	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
	49·3	52	55	57·2	58·4	60·5	61·4	62·5	62·9	62·2	62·5	61·1	59·8	58·2	55·8

	27	28	29	30	31	32	33	34	35	36
	54·2	51	48·8	46·8	43·5	42·1	40·6	39·8	38·8	38·6

13. s is the displacement from a fixed point, of a tramcar, in time t secs. Draw the space-time, velocity-time and acceleration-time curves.

t	0	1	2	3	4	5	6	7	8	9	10
s	0	4	11	21	34	50	69	91	116	144	175

The scales must be clearly indicated.

14. The table gives the temperature of a body at time t secs. after it has been left to cool. Plot the given values and thence by differentiation obtain the rate of cooling curve. What conclusions do you draw from your final curve?

Time (mins.)	0	1	2	3	4	5	6	7	8	9
Temp. (F.°)	136	134	132	130	128	126·5	124·8	123·3	122	120·5

	10	11	12	13	14	15	16	17	18	19
	119·3	118	116·8	115·5	114·5	113·5	112·5	111·5	110·5	109·5

15. The following figures give the bending moment at various points along a beam supported at both ends and loaded uniformly. Draw the bending moment curve, and by graphic differentiation obtain the shear and load curves. Indicate clearly the scales and write down the value of the load per foot run.

Distance from one end (ft.) . .	0	2	4	6	8	10	12	14	16	18	20
Bending moment (tons ft.) . .	0	3·5	6·3	8·4	9·6	10	9·6	8·4	6·3	3·5	0

16. By taking values of θ in the neighbourhood of $15°$ find the actual rate of change of $\sin\theta$ with regard to θ (θ being expressed in radians). Compare your result with the value of $\cos 15°$. In what general way could the result be expressed?

17. If the shear at various points in the length of a beam is as in the table, draw the load curve (*i. e.*, the derived curve) and write down the loading at $3\frac{1}{2}$ ft. from the left-hand end.

Distance from left-hand end (ft.)	0	1	2	3	4	5	6
Shearing force (tons)	0	·1	·3	·6	1	1·5	2·1

18. An E.M.F. wave is given by the equation

$$E = 150 \sin 314t + 50 \sin 942t.$$

Derive graphically the wave form of the current which the E.M.F. will send through a capacitor of 20 microfarads capacitance, assuming the capacitor loss to be negligible.

Given that $I = C\dfrac{dE}{dt}$, where I is current and C is capacitance.

19. If momentum is given by the product of mass into velocity, and force is defined as the time rate of change of momentum, show that force is expressed by the product of mass into acceleration.

20. The following are the approximate speeds of a locomotive on a run over a not very level road. Plot these figures and thence obtain a curve showing the acceleration at any time during the run.

Time (in mins. and secs.)	0	1·0	2·15	6·15	9·22	11·45	14·26	16·35	20·52	23·10
Speed (miles per hour) .	start	6	10	18·2	22·8	25·5	28	29·2	28·6	26·1

21. Taking the following figures referring to CO_2 for use in a refrigerating machine, draw the rate curve and find the value of $\dfrac{dp}{dt}$ when $t = 18°$ F.

$t°$ F. .	−5	0	5	10	15	20	25	30	35	40
p lbs. per sq. in	285	310	335	363	392	423	456	491	528	567

22. The weight of a sample of cast iron was measured after various heatings with the following results; the gain in weight being due to the external gases in the muffle.

Number of heats	0	2	6	12	22	23	24	25	26	27	30	35	39	45
Weight	146·88	146·94	147·04	147·54	148·02	148·11	148·27	148·36	148·46	148·61	149·18	150·49	152·36	156·44

Plot a curve to represent this table of values, and from it construct the rate curve.

23. The figures in the table are the readings of the temperature of a sample of steel at various times during its cooling. Plot these values to a time base, and thence draw the "inverse rate" curve, *i.e.*, the curve in which values of $\frac{dt}{d\theta}$ are plotted horizontally and the temperatures along the vertical axis.

Time in secs. (t)	Temperature in C.° (θ)
75	850
90	848
105	844·7
120	842
135	839·5
150	838·5
165	838·2
180	838·1
195	838
210	837·9
225	837·5
240	836
255	833
270	829
285	825
292·5	823·3
300	822·2
315	821·7
330	821·5
345	821·3
360	821·1
367·5	819
375	815
390	813
405	811·6

24. Plot the curve $y = 4x^2 - 11x - 18$ from $x = -4$ to $x = 6$, and from it obtain its slope curve. What is the equation to the slope curve?

CHAPTER II

DIFFERENTIATION OF FUNCTIONS

Differentiation of ax^n.—It has been shown in Chapter I how to compare the changes in two quantities with one another, and thus to determine the rate at which one is changing with regard to the other at any particular instant, for cases in which sets of values of the two variables have been stated. In a great number of instances, however, the two quantities are connected by an equation, indicating that the one depends upon the other, or, in other words, one is a *function* of the other. Thus if $y = 5x^3$, y has a definite value for each value given to x, and this fact is expressed in the shorter form $y = f(x)$. Again, if $z = 17x^2y - 4xy^3 + 5 \log y$, where both x and y vary, z depends for its values on those given to both x and y, and $z = f(x, y)$.

To differentiate a function it is not necessary to calculate values of x and y and then to treat them as was done to the given sets of values in the previous chapter. This would occasion a great waste of time and would not give absolutely accurate results. Rules can be developed entirely from first principles which permit the differentiation of functions without any recourse to tables of values or to a graph.

We now proceed to develop the first of the rules for the differentiation of functions; and we shall approach the general case, viz., that of $y = ax^n$, by first considering the simple case of $y = x^3$. Our problem is thus to find the rate at which y changes with regard to x, the two variables (y the dependent and x the independent variable or I.V.) being connected by the equation $y = x^3$.

The rate of change of y with regard to x is given by the value of $\frac{dy}{dx}$, and this is sometimes written as Dy when it is clearly understood that differentiation is with regard to x: the *operator* D having many important properties, as will be seen later in the book. If y is expressed as $f(x)$, then $\frac{dy}{dx}$ is often written $\frac{df(x)}{dx}$ or $f'(x)$.

$\frac{dy}{dx}$, Dy, $\frac{df(x)}{dx}$ or $f'(x)$ is called the *derivative* or *differential coefficient* of y with respect to x; and the full significance of the latter of these terms is shown in Chapter III.

We wish to find a rule giving the actual rate of change of y with regard to x, y being $= x^3$, the rule to be true for all values of x. As in the earlier work, the actual rate of change must be determined as the limiting value of the average rate of change.

Let x be altered by an amount δx so that the new value of $x = x + \delta x$; then y, which depends upon x, must change to a new value $y + \delta y$, and since the relation between y and x is $y = x^3$ for *all* values of x—

$$(\text{new value of } y) = (\text{new value of } x)^3$$

$$\text{or} \quad y+\delta y = (x+\delta x)^3 = x^3+3x^2 . \delta x+3x . (\delta x)^2+(\delta x)^3 \quad (1)$$

$$\text{but} \quad y = x^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, by subtraction of (2) from (1)—

$$y+\delta y-y = 3x^2 . \delta x+3x(\delta x)^2+(\delta x)^3$$

$$\text{and} \quad \delta y = 3x^2 . \delta x+3x(\delta x)^2+(\delta x)^3.$$

Divide through by δx, and—

$$\frac{\delta y}{\delta x} = 3x^2+3x . \delta x+(\delta x)^2.$$

Thus an *average* value for the rate of change over a small interval δx has been found; and to deduce the *actual* rate of change the interval δx must be reduced indefinitely.

Let $\delta x = \cdot 001$; then $\frac{\delta y}{\delta x} = 3x^2+(3x\times\cdot 001)+\cdot 000001$

$$= 3x^2+\cdot 003x+\cdot 000001$$

whilst if $\delta x = \cdot 00001$—

$$\frac{\delta y}{\delta x} = 3x^2+\cdot 00003x+\cdot 0000000001 \quad . \quad (3)$$

Evidently, by still further reducing δx the 2nd and 3rd terms of (3) can be made practically negligible in comparison with the 1st term.

Then, in the limit, the right-hand side becomes $3x^2$,

$$\text{and thus—} \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 3x^2.$$

$$\text{or} \quad \frac{dx^3}{dx} = 3x^2.$$

This relation can be interpreted graphically in the following manner: If the curve $y = x^3$ be plotted, and if also its slope curve be drawn by either of the methods of Chapter I, then the equation to the latter curve is found to be $y = 3x^2$.

The two curves are plotted in Fig. 7.

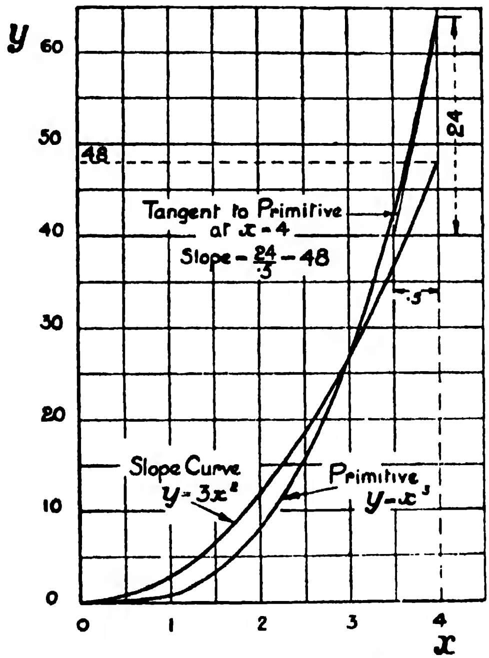


FIG. 7.—Primitive and Slope Curves.

Example 1.—Find the slope of the curve $y = x^3$ when $x = 4$.

$$\text{The slope of the curve} = \frac{dy}{dx} = \frac{dx^3}{dx} = 3x^2$$

$$\text{and if } x = 4 \qquad \frac{dy}{dx} = 3 \times 4^2 = \underline{48.}$$

Meaning that, in the neighbourhood of $x = 4$, the ordinate of the curve $y = x^3$ is changing 48 times as fast as the abscissa; this fact being illustrated by Fig. 7.

Working along the same lines, it would be found that $\frac{dx^4}{dx} = 4x^3$, and $\frac{dx^5}{dx} = 5x^4$ (the reader is advised to test these results for himself).

Re-stating these relations in a modified form—

$$\frac{dx^3}{dx} = 3x^2 = 3x^{3-1}$$

$$\frac{dx^4}{dx} \quad 4x^3 = 4x^{4-1}$$

$$\frac{dx^5}{dx} = 5x^4 = 5x^{5-1}$$

We note that in all these cases the results take the form—

$$\frac{dx^n}{dx} = nx^{n-1}.$$

Thus the three cases considered suggest a general rule, but it would be unwise to accept this as the true rule without the more rigid proof, which can now be given.

Proof of the rule—

$$\frac{dx^n}{dx} = nx^{n-1}.$$

Let $y = x^n$, this relation being true for all values of x . . (1)

If x is increased to $x+\delta x$, y takes a new value $y+\delta y$, and from (1) it is seen that—

$$y+\delta y = (x+\delta x)^n.$$

Expand $(x+\delta x)^n$ by the Binomial Theorem (see p. 463, Part I). Then—

$$y+\delta y = x^n+nx^{n-1}\delta x+\frac{n(n-1)}{\lfloor\underline{2}}x^{n-2}(\delta x)^2$$
$$+\frac{n(n-1)(n-2)}{\lfloor\underline{3}}x^{n-3}(\delta x)^3+\ \ldots \qquad . \ . \ . \quad (2)$$

Subtract (1) from (2), and—

$$\delta y = nx^{n-1}\delta x+\frac{n(n-1)}{\lfloor\underline{2}}x^{n-2}(\delta x)^2+\frac{n(n-1)(n-2)}{\lfloor\underline{3}}x^{n-3}(\delta x)^3+\ \ldots$$

Divide by δx—

$$\frac{\delta y}{\delta x} = nx^{n-1}+\frac{n(n-1)}{\lfloor\underline{2}}x^{n-2}(\delta x)+\frac{n(n-1)(n-2)}{\lfloor\underline{3}}x^{n-3}(\delta x)^2+\text{terms}$$

containing products of $(\delta x)^3$ and higher powers of (δx).

Let δx be continually decreased, and then, since δx is a factor of the second and all succeeding terms, the values of these terms can be made as small as we please by sufficiently diminishing δx.

Thus in the limit— $\dfrac{\delta y}{\delta x} \to nx^{n-1}$

or $$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = nx^{n-1}$$

Hence the first rule for differentiation of functions is established, viz.—

$$\frac{dx^n}{dx} = nx^{n-1}$$

i. e., differentiation lowers the power of the I.V. by one, but the new power of x must be multiplied by the original exponent.

The reason for the multiplication by the n can be readily seen, for the bigger the value of n the steeper is the primitive curve and therefore the greater the change of y for unit change of x. The n actually determines the slope of the primitive (cf. Part I, p. 340), and it must therefore be an important factor in the result of differentiation, since that operation gives the equation of the slope curve.

To make the rule perfectly general, allowance must be made for the presence of the constant multiplier a in ax^n.

It will be agreed that if the curve $y = x^3$ had been plotted, the curve $y = 4x^3$ would be the same curve modified by simply multiplying the vertical scale by 4. Hence, in the measurement of the slope, the vertical increases would be four times as great for the curve $y = 4x^3$ as for the curve $y = x^3$, provided that the same horizontal increments were considered.

Now the slope of the curve $y = x^3$ is given by the equation—

$$\frac{dy}{dx} = 3x^2$$

so that the slope of the curve $y = 4x^3$ is given by—

$$\frac{dy}{dx} = 4 \times 3x^2 = 12x^2.$$

In other words, the constant multiplier 4 remains a multiplier throughout differentiation. This being true for any constant factor—

$$\frac{d}{dx}ax^n = nax^{n-1}.$$

Accordingly, a constant factor before differentiation remains as such after differentiation.

We can approach the differentiation of a multinomial expression

by discussing the simple case $y = 5x^2 + 17$ (a binomial, or two-term expression). The curves $y = 5x^2$ and $y = 5x^2 + 17$ are seen plotted in Fig. 8, and an examination shows that the latter curve is the former moved vertically an amount equal to 17 vertical units, *i. e.*, the two curves have the same form or shape and consequently their slopes at corresponding points are alike. Thus if a tangent is drawn to each curve at the point for which $x = 2{\cdot}5$, the slope of each tangent is measured as $\frac{25}{1}$, *i. e.*, 25; and consequently the diagram informs us that the term 17 makes no difference to the slope.

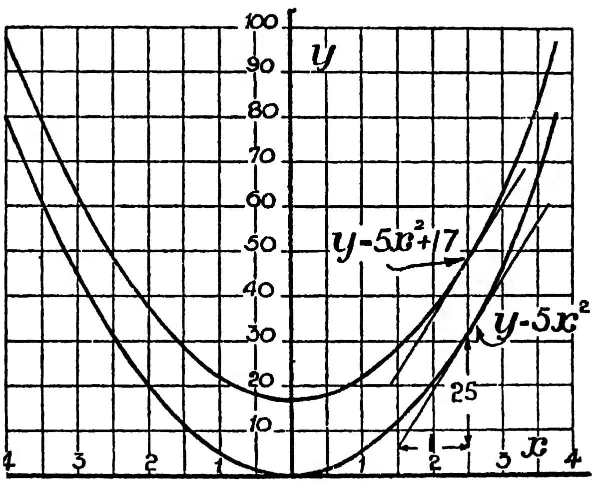


FIG. 8.

Thus— $$\frac{d}{dx}5x^2 = \frac{d}{dx}(5x^2 + 17).$$

Now, by differentiating $5x^2 + 17$ term by term, we have—

$$\frac{d}{dx}(5x^2+17) = \frac{d}{dx}5x^2+\frac{d}{dx}17 = \frac{d}{dx}5x^2+0$$
$$= \frac{d}{dx}5x^2$$

since 17 is a constant and does not in any way depend upon x, and therefore its rate of change must be zero.

It is seen that in this simple example it is a perfectly logical procedure to differentiate term by term and then add the results; and the method could be equally well applied to all many-term expressions.

Hence— $\frac{d}{dx}(ax^n+bx^{n-1}+cx^{n-2}+\ldots d)$

$$= nax^{n-1}+b(n-1)x^{n-2}+c(n-2)x^{n-3}+\ldots$$

and $$\frac{d}{dx}(ax^n+b) = nax^{n-1}.$$

To apply these rules to various numerical examples :—

Example 2.—Differentiate with respect to x the function—

$$9x^{1\cdot6}+4+2x^{\cdot5}.$$

$$\frac{d}{dx}(9x^{1\cdot6}+4+2x^{\cdot5}) = \frac{d}{dx}9x^{1\cdot6}+\frac{d}{dx}4+\frac{d}{dx}2x^{\cdot5}$$
$$= (9\times1\cdot6x^{\cdot6})+0+(2\times\cdot5x^{-\cdot5})$$
$$= 14\cdot4x^{\cdot6}+x^{-\cdot5} \quad\text{or}\quad \underline{14\cdot4x^{\cdot6}+\frac{1}{\sqrt{x}}}$$

Example 3.—If $y = \cdot8\times\sqrt{\frac{1}{x^3}}$, find the value of $\frac{dy}{dx}$.

$$y = \cdot8\times\sqrt{\frac{1}{x^3}} = \cdot8\times\frac{1}{x^{\frac{3}{2}}} = \cdot8x^{-\frac{3}{2}} \quad\text{or}\quad \cdot8x^{-1\cdot5}$$

so that in comparison with the standard form—

$$a = \cdot8 \text{ and } n = -1\cdot5.$$

Then—

$$\frac{dy}{dx} = -1\cdot5\times\cdot8x^{-1\cdot5-1} = -1\cdot2x^{-2\cdot5} \quad\text{or}\quad \underline{-\frac{1\cdot2}{x^{2\cdot5}}}$$

Example 4.—If $pv^{1\cdot41} = C$, the equation representing the adiabatic expansion of air, find $\frac{dp}{dv}$.

In this example we have to differentiate p with regard to v, and before this can be done p must be expressed in terms of v.

Now $pv^{1\cdot41} = C$, so that $p = \frac{C}{v^{1\cdot41}} = Cv^{-1\cdot41}$.

Hence $$\frac{dp}{dv} = \frac{d}{dv}Cv^{-1\cdot41} = C\times-1\cdot41v^{-2\cdot41} = -1\cdot41Cv^{-2\cdot41}$$

and this result can be put into terms of p and v only, if for C we write its value $pv^{1\cdot41}$.

Thus $$\frac{dp}{dv} = -1\cdot41\times pv^{1\cdot41}\times v^{-2\cdot41} = -1\cdot41pv^{-1} = -\frac{1\cdot41p}{v}.$$

Example 5.—The formula giving the electrical resistance of a length of wire at temperature t° C. is—

$$R_t = R_0(1+\alpha t)$$

where R_0 = Resistance at 0° C. Find the increase of resistance per 1° C. rise of temperature per ohm of initial resistance, and hence state a meaning for α.

The question may be approached from two standpoints; viz.—

(*a*) Working from first principles.

$$R_t = R_0(1+\alpha t) = R_0+R_0\alpha t$$

$$R_t - R_0 = R_0\alpha t$$

i. e., increase in resistance for t° C. $= R_0\alpha t$

" " " " 1° C. $= \dfrac{R_0\alpha t}{t} = R_0\alpha$

but this is the resistance increase for initial resistance R_0, hence increase in resistance per 1° C. per ohm initial resistance $= \dfrac{R_0\alpha}{R_0} = \alpha$.

(*b*) By differentiation.

Rate of change of R_t with regard to $t = \dfrac{dR_t}{dt}$

$$= \frac{d}{dt}(R_0+R_0\alpha t)$$

$$= 0+R_0\alpha = R_0\alpha$$

and consequently the rate of change of resistance per 1° C. per 1 ohm initial resistance $= \alpha$.

The symbol α is thus the "temperature coefficient," its numerical value for pure metals being ·0038.

Example 6.—Find the value of $\dfrac{d}{ds}\left(4s^4-\dfrac{3}{s^2}+6s^{\frac{1}{2}}-1{\cdot}8^4\right)$.

Write the expression as $4s^4-3s^{-2}+6s^{\cdot 5}-1{\cdot}8^4$.

Then—

$$\frac{d}{ds}(4s^4-3s^{-2}+6s^{\cdot 5}-1{\cdot}8^4) = 4\times 4s^3-(3\times -2s^{-3})+(6\times{\cdot}5s^{-\cdot 5})-0$$

$$= 16s^3+6s^{-3}+3s^{-\cdot 5}$$

$$= \underline{16s^3+\frac{6}{s^3}+\frac{3}{\sqrt{s}}}$$

Example 7.—If $x = a^{\frac{2}{n}}\left(1-a^{\frac{n-1}{n}}\right)$, a formula referring to the flow of a gas through an orifice, find an expression for $\dfrac{dx}{da}$.

As it stands $a^{\frac{2}{n}}\left(1-a^{\frac{n-1}{n}}\right)$ is a product of functions of the I.V. (in this case a), and it cannot therefore be differentiated with our

present knowledge. We may simplify, however, by removing the brackets, and then—

$$x = a^{\frac{2}{n}} - a^{\frac{n-1}{n}+\frac{2}{n}} = a^{\frac{2}{n}} - a^{\frac{n+1}{n}}$$

$$\frac{dx}{da} = \frac{d}{da}\left(a^{\frac{2}{n}} - a^{\frac{n+1}{n}}\right)$$

$$= \frac{2}{n} \times a^{\frac{2}{n}-1} - \frac{n+1}{n} \cdot a^{\frac{n+1}{n}-1}$$

$$= \frac{2}{n} a^{\frac{2-n}{n}} - \frac{n+1}{n} a^{\frac{1}{n}}$$

$$= \underline{\frac{1}{n}\left(2a^{\frac{2-n}{n}} - (n+1)a^{\frac{1}{n}}\right).}$$

Example 8.—Determine the value of—

$$\frac{d}{dm}\left(\frac{17m^{\cdot 75} - \cdot 45m^{9\cdot 86} + 11 + 2\cdot 4m^{4\cdot 32}}{5m^{4\cdot 32}}\right).$$

To avoid the quotient of functions of m, divide each term by $5m^{4\cdot 32}$,

then the expression $= \frac{17m^{\cdot 75}}{5m^{4\cdot 32}} - \frac{\cdot 45m^{9\cdot 86}}{5m^{4\cdot 32}} + \frac{11}{5m^{4\cdot 32}} + \frac{2\cdot 4m^{4\cdot 32}}{5m^{4\cdot 32}}$

$= 3\cdot 4m^{-3\cdot 57} - \cdot 09m^{5\cdot 54} + 2\cdot 2m^{-4\cdot 32} + \cdot 48$

and $\frac{d}{dm}$ (expression) $= (3\cdot 4 \times -3\cdot 57m^{-4\cdot 57}) - (\cdot 09 \times 5\cdot 54m^{4\cdot 54})$

$+ (2\cdot 2 \times -4\cdot 32m^{-5\cdot 32}) + 0$

$= -12\cdot 14m^{-4\cdot 57} - \cdot 499m^{4\cdot 54} - 9\cdot 50m^{-5\cdot 32}$

$= \underline{-\left(\frac{12\cdot 14}{m^{4\cdot 57}} + \cdot 499m^{4\cdot 54} + \frac{9\cdot 5}{m^{5\cdot 32}}\right).}$

Proof of the construction for the slope curve given on p. 14.

Let us deal first with the particular case in which the equation of the primitive curve is $y = x^2$.

Referring to Fig. 4, the equation of the curve OAB is $y = x^2$, and the equation of the curve CG is $y_1 = (x - h)^2 = x^2 + h^2 - 2xh$.

Hence the difference between the ordinates of the curves OAB and CG, the latter being regarded as the base curve—

$$= x^2 - x^2 - h^2 + 2xh$$
$$= 2xh - h^2$$

so that the equation of the curve MN is—

$$y_2 = 2xh - h^2.$$

Now the curve ODE is the curve MN shifted back a distance of $\frac{h}{2}$ horizontal units, and hence its equation is $y_3 = 2\left(x+\frac{h}{2}\right)h - h^2$, since for x we must now write $\left(x+\frac{h}{2}\right)$.

Thus the equation of curve ODE is—

$$y_3 = 2xh$$

or

$$\frac{y_3}{h} = 2x$$

i. e., if Y be written for $\frac{y_3}{h}$, $Y = 2x$

or the equation of the curve ODE is that of the slope curve of the curve $y = x^2$ provided that the ordinates are read to a certain scale; this scale being the original vertical scale divided by h expressed in horizontal units.

Hence the curve ODE is the slope curve of the curve OAB.

Before discussing the general case, let us take the case of the primitive with equation $y = x^3$.

If the curve be shifted forward an amount $= h$, the equation of the new curve is—

$$y_1 = (x-h)^3$$

and the equation of the curve giving the differences of the ordinates is—

$$\begin{aligned} y_2 = y-y_1 = x^3-(x-h)^3 &= x^3-x^3+3x^2h-3xh^2+h^3 \\ &= 3x^2h-3xh^2+h^3. \end{aligned}$$

By shifting this curve $\frac{h}{2}$ units to the left we change its equation, by writing $\left(x+\frac{h}{2}\right)$ in place of x, into the form—

$$\begin{aligned} y_3 &= 3\left(x+\frac{h}{2}\right)^2 h-3\left(x+\frac{h}{2}\right)h^2+h^3 \\ &= 3x^2h+\frac{3h^3}{4}+3h^2x-3xh^2-\frac{3h^3}{2}+h^3 \\ &= 3x^2h+\frac{h^3}{4} \end{aligned}$$

Dividing by h—

$$\frac{y_3}{h} = 3x^2+\frac{h^2}{4}$$

or

$$Y = 3x^2+\frac{h^2}{4}$$

Now if h is taken sufficiently small, $\frac{h^2}{4}$ is negligible in comparison with $3x^2$, and we thus have the equation of the curve $Y = 3x^2$, which is the slope curve of the curve $y = x^3$; but the ordinates must be measured to the old vertical scale divided by h.

We may now consider the case of the primitive $y = x^n$. Adopting the notation of the previous illustrations—

$$y_1 = (x-h)^n$$

$$y_2 = y - y_1 = x^n - (x-h)^n = x^n - \left(x^n - nx^{n-1}h + \frac{n(n-1)}{\underline{|2}}x^{n-2}h^2 - \ldots\right)$$

$$= nx^{n-1}h - \frac{n(n-1)}{\underline{|2}}x^{n-2}h^2 + \ldots$$

Write $\left(x+\frac{h}{2}\right)$ in place of x, and then—

$$y_2 = n\left(x+\frac{h}{2}\right)^{n-1}h - \frac{n(n-1)}{\underline{|2}}\left(x+\frac{h}{2}\right)^{n-2}h^2 + \ldots$$

$$\frac{y_2}{h} = \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \frac{n(n-1)(n-2)}{8}x^{n-3}h^2 + \ldots\right.$$

$$-\frac{n(n-1)}{2}x^{n-2}h - \frac{n(n-1)(n-2)}{4}x^{n-3}h^2 - \ldots$$

$= nx^{n-1} +$ terms containing h as a factor.

Hence if h is made very small—

$$Y \text{ or } \frac{y_2}{h} = nx^{n-1}.$$

Exercises 2.—On Differentiation of Powers of the I.V.

1. Find from first principles the differential coefficient of x^4

2. Find the slope of the curve $y = \frac{8}{x^2}$ when $x = \cdot 5$

   (*a*) By actual measurement and (*b*) by differentiation.

3. The sensitiveness of a governor is measured by the change of height corresponding to the change of speed expressed as a fraction of the speed. Thus if h and v represent respectively the height and speed, the sensitiveness $= dh \div \frac{dv}{v}$. If the height is inversely proportional to the square of the velocity, find an expression for the sensitiveness.

Differentiate with respect to x the functions in Exs. 4 to 15.

4. $3x^9$. 5. $\frac{15}{x^5}$. 6. $81\cdot 5x^{-\cdot 23}$. 7. $19x^{1\cdot 72}$. 8. $\frac{\cdot 215}{4x^{-\cdot 16}}$.

9. $\dfrac{8x^{4.13}}{2{\cdot}94x^{-7.85}}$. 10. $5\sqrt[9]{x^{-2}}$. 11. $\left(\dfrac{4x^{-2}}{7x^{\frac{1}{2}}}\right)^{\frac{1}{3}}$. 12. $\dfrac{(x^{3.7})^{2.9}}{15x^{1.42}}$.

13. $15x^2-16x^{2.3}-\dfrac{14}{2x^3}+87$. 14. $9y^{3.17}$. 15. $\dfrac{8a^2b^{1.75}}{7\cdot\sqrt[3]{27x^{1.5}}}$.

16. Find the value of $\dfrac{dp}{dv}$ when $pv^{1.3} = 570$ and $v = 28{\cdot}1$.

17. Find the value of $\dfrac{d}{dv}\left(\dfrac{1{\cdot}17v^9-2v^{.8}-3v^{.72}+v^{-.04}+24}{8v^{4.16}}\right)$.

18. If $E = -15+14T - {\cdot}0068T^2$, find the rate of change of E with regard to T when T has the value 240.

19. Calculate the value of $\dfrac{dH}{dv}$ from $\dfrac{dH}{dv} = \dfrac{1}{\gamma-1}\left\{v\dfrac{dp}{dv}+\gamma p\right\}$ when $pv^{1.3} = C$ and $\gamma = 1{\cdot}4$.

20. Find the rate of discharge $\left(\dfrac{dm}{dt}\right)$ of air through an orifice from a tank (the pressure being 55 lbs./$\square''$) from the following data—

$$144pV = mRT$$
$$R = 53{\cdot}2,\ V = 47{\cdot}7,\ T = 548.$$

Time (secs.) (t)	0	60	135	255	315
Pressure (lbs. per sq. in.) (p)	63	45	30	15	10

Hint.—Plot p against t and find $\dfrac{dp}{dt}$ when $p = 55$.

21. If P = load displacement of a ship,
p = weight of offensive and defensive factors.
Then $P = aP+bP^{\frac{2}{3}}+p$.
Find the rate of increase of armament and protection in relation to increase of displacement.

22. If $M = \dfrac{w(l-x)^2y}{2l}\left(1+\dfrac{y}{l}\right)-\dfrac{w(y-x)^2}{2}$, find $\dfrac{dM}{dx}$, y and l being constants.

23. If $M = \dfrac{Wy}{2l^2}(l^2 - 4y^2)$, find the value of y that makes $\dfrac{dM}{dy} = 0$.

24. If $S = \dfrac{w}{2}\left\{\dfrac{(x+ny)^2}{l}-\dfrac{x^2}{y}\right\}$, find the value of $\dfrac{dS}{dx}$.

25. Find the value of h which makes $\dfrac{dD}{dh} = 0$ when—

$$D = \frac{f_c+f_t}{2Eh}\left\{\frac{l^2}{4}+\frac{ld}{4}+\frac{lh^2}{d}\right\}.$$

[h is the height of a Warren girder; and the value found will be the height for maximum stiffness.]

26. If $p = \dfrac{2B}{r^3}-A$ and $q = \dfrac{B}{r^3}+A$, find the value of $r\dfrac{dp}{dr}$ in terms of p and q. (This question refers to the stresses in a thick spherical shell, p being the radial pressure, and q the hoop tension.)

27. In a certain vapour the relation between the absolute temperature τ and the absolute pressure p is given by the equation $\tau = 140p^{\frac{1}{4}}+465$, and the latent heat L is given by $L = 1431 - \cdot 5\tau$. Find the volume, in cu. ft., of 1 lb. of the vapour when at a pressure of 81 lbs. per sq. in. absolute, from—

$$V - \cdot 02 = \frac{JL}{144\tau} \cdot \frac{d\tau}{dp} \qquad \{J = 778\}.$$

28. For a rolling uniform load of length r on a beam of length l, the bending moment M at a point is given by—

$$M = \frac{wry}{l}\left(l-y+x-\frac{r}{2}\right)-\frac{wx^2}{2}.$$

If y is a constant, find an expression for the shear (*i. e.*, the rate of change of bending moment).

29. Given that ρ = electrical resistance in microhms per cu. cm.
and x = percentage of aluminium in the steel,
then $\rho = 12+12x-\cdot 3x^2$ for steel with low carbon content.
Find the rate of increase of ρ with increase of aluminium when $x = 4$.

30. The equation giving the form taken by a trolley wire is—

$$y = \frac{x^2}{2000} - \frac{x^2}{1760}$$

and the radius of curvature $= \dfrac{1}{\dfrac{d^2y}{dx^2}}$

Find the value of the radius of curvature.

Good examples of the great advantage obtained by utilising the rules of differentiation already proved are furnished by the two following examples, which have reference to loaded beams.

Example 9.—Prove that the shearing force at any point in a beam is given by the rate of change of the bending moment at that point.

Consider two sections of the beam δx apart (see Fig. 9). The shear at a section being defined as the sum of all the force to the right of that section, let the shear at b = S, and let the shear at a = S+δS. Also let the moment of all the forces to the right of b (*i. e.*, the bending moment at b) = M, and let the bending moment at a = M+δM.

Taking moments about C—

$$M+\delta M = M+(S+\delta S)\frac{\delta x}{2}+S\left(\frac{\delta x}{2}\right)$$

$$= M+S\,.\,\delta x+\frac{\delta S\,.\,\delta x}{2}$$

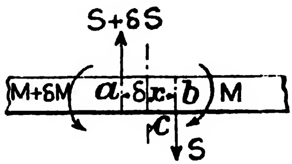
or

$$\delta M = S\delta x+\frac{\delta S\,.\,\delta x}{2}.$$

Dividing by δx, $$\frac{\delta M}{\delta x} = S + \frac{\delta S}{2}$$

and when δx is diminished indefinitely, δS becomes negligible

and $$\frac{dM}{dx} = S.$$



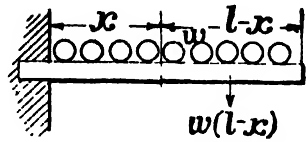


FIG. 9. FIG. 10.

Examples on Loaded Beams.

The last example should be considered in conjunction with the following :—

Example 10.—For a beam of length l, fixed at one end and loaded uniformly with w tons per foot run, the deflection y at distance x from the fixed end is given by the formula—

$$y = \frac{w}{24EI}\{6l^2x^2 - 4lx^3 + x^4\}$$

E being the Young's Modulus of the material of the beam, and I being the moment of inertia of the beam section.

Find the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$.

$$y = \frac{w}{24EI}\{6l^2x^2 - 4lx^3 + x^4\}$$

Differentiating, $$\frac{dy}{dx} = \frac{w}{24EI}\{(6l^2 \times 2x) - (4l \times 3x^2) + 4x^3\}$$

$$= \frac{w}{24EI}\{12l^2x - 12lx^2 + 4x^3\}$$

$$= \frac{w}{6EI}\{3l^2x - 3lx^2 + x^3\}.$$

Differentiating again, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{w}{6EI}\{3l^2x - 3lx^2 + x^3\}\right]$$

$$= \frac{w}{6EI}\{3l^2 - 6lx + 3x^2\}$$

$$= \frac{w}{2EI}\{l^2 - 2lx + x^2\}$$

or $$\frac{w}{2EI}(l-x)^2$$

Differentiating again, $\dfrac{d^3y}{dx^3} = \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{d}{dx}\left[\dfrac{w}{2\mathrm{EI}}(l^2-2lx+x^2)\right]$

$$= \frac{w}{2\mathrm{EI}}(0-2l+2x)$$

$$= \underline{\frac{w}{\mathrm{EI}}(x-l).}$$

Carrying the differentiation one stage further—

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d}{dx}\left[\frac{w}{\mathrm{EI}}(x-l)\right]$$

$$= \frac{w}{\mathrm{EI}}(1-0) = \underline{\frac{w}{\mathrm{EI}}}.$$

Physical meanings may now be found for these various derivatives. Referring to Fig. 10, consider a section of the beam distant x from the fixed end. To the right of this section there is a length of beam $l-x$ loaded with w tons per foot, so that the total load or total downward force on this length is $w(l-x)$; and since this load is evenly distributed, it may be all supposed to be concentrated at distance $\dfrac{l-x}{2}$ from the section.

Now the bending moment at the section

= moment of all the force to the right of the section

$= \text{force} \times \text{distance} = w(l-x)\times\left(\dfrac{l-x}{2}\right) = \dfrac{w}{2}(l-x)^2.$

Comparing this result with the value found for $\dfrac{d^2y}{dx^2}$, we notice that the two are alike except for the presence of the constants E and I: thus $\dfrac{d^2y}{dx^2}$ must be a measure of the bending moment.

Actually the rule connecting M, the bending moment, and $\dfrac{d^2y}{dx^2}$ is—

$$\frac{\mathrm{M}}{\mathrm{I}} = \mathrm{E}\frac{d^2y}{dx^2} \quad \text{or} \quad \mathrm{M} = \mathrm{EI}\frac{d^2y}{dx^2}$$

the proof of this rule being given in a later chapter.

Again, we have proved in the previous example that the shear is given by the rate of change of bending moment: thus—

$$\mathrm{S} = \frac{d\mathrm{M}}{dx} = \frac{d}{dx}\left(\mathrm{EI}\frac{d^2y}{dx^2}\right) = \mathrm{EI}\frac{d^3y}{dx^3}$$

$$= \mathrm{EI}\times\frac{w}{\mathrm{EI}}(x-l)$$

$$= w(x-l)$$

a result agreeing with our statement that the shear at a section is the sum of all the loads to the right of the section. [The reason for the minus sign, viz., $(x-l)$, being written where $(l-x)$ might be expected need not be discussed at this stage.]

Continuing the investigation—

$$\frac{d^4y}{dx^4}=\frac{w}{EI}$$

or

$$EI\frac{d^4y}{dx^4}=w$$

but w is the loading on the beam and—

$$EI\frac{d^4y}{dx^4}=\frac{d}{dx}\left(EI\frac{d^3y}{dx^3}\right)=\frac{dS}{dx}$$

so that the loading is measured by the rate of change of the shear.

If now the deflected form is set out, by constructing successive slope curves we obtain, respectively, the slope curve of the deflected form, the bending moment curve, the shear curve and finally the curve of loads.

Example 11.—The work done in the expansion of gas in gas turbines is given by—

$$W=\frac{n}{n-1}P_1V_0\frac{T_1}{T_0}\left(1-r^{\frac{n-1}{n}}\right)$$

where r is the ratio of expansion.

Compare governing by expansion control with governing by alteration of the initial temperature, from the point of view of efficiency.

Deal first with the expansion control, *i. e.*, regard T_1 as constant and r as variable. Then the rate at which the work is increased with respect to r is $\frac{dW}{dr}$.

Now
$$\frac{dW}{dr}=\frac{d}{dr}\left[\frac{n}{n-1}P_1V_0\frac{T_1}{T_0}\left(1-r^{\frac{n-1}{n}}\right)\right]$$

$$=\frac{n}{n-1}P_1V_0\frac{T_1}{T_0}\left(0-\frac{n-1}{n}r^{\frac{n-1}{n}-1}\right)$$

$$=-\frac{n}{n-1}P_1V_0\frac{T_1}{T_0}\times\frac{n-1}{n}r^{-\frac{1}{n}}$$

$$=-\frac{P_1V_0T_1}{T_0r^{\frac{1}{n}}}$$

Now regard r as constant, but T_1 as variable.

Then—
$$\frac{dW}{dT_1} = \frac{n}{n-1}\frac{P_1V_0}{T_0}\left(1 - r^{\frac{n-1}{n}}\right)$$

and, expressing the two results in the form of a ratio—
$$\frac{dW}{dr} \div \frac{dW}{dT_1} = -\frac{P_1V_0T_1}{T_0r^{\frac{1}{n}}} \times \frac{(n-1)T_0}{nP_1V_0\left(1 - r^{\frac{n-1}{n}}\right)}$$
$$= \underline{\frac{(1-n)T_1}{nr^{\frac{1}{n}}\left(1 - r^{\frac{n-1}{n}}\right)}}$$

Lengths of Sub-tangents and Sub-normals of Curves.—The projection of the tangent to a curve on to the axis of x is known as the *sub-tangent, i. e.*, the distance "sub" or "under" the tangent. The projection of the normal on the x axis is called the *sub-normal.*

The slope of a curve at any point, measured by the slope of its tangent at that point, is given by the value of $\frac{dy}{dx}$ there, or if α = inclination of the tangent to the positive direction of the x axis—
$$\tan \alpha = \frac{dy}{dx}$$

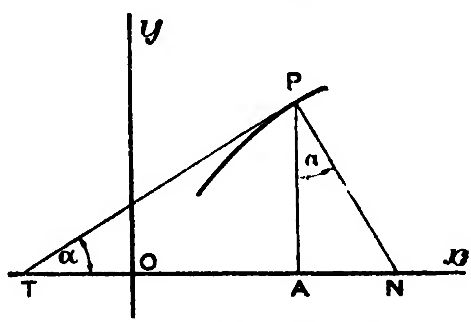


FIG. 11.—Sub-tangent and Sub-normal.

In Fig. 11—
$$\frac{PA}{AT} = \tan \alpha = \frac{dy}{dx}$$
$$\therefore \quad AT = PA\frac{dx}{dy}$$

But- AT = sub-tangent and PA = y

and hence the length of the sub-tangent $= y\frac{dx}{dy}$

Again— $\angle APN = \alpha$, since $\angle TPN =$ a right angle

$$\tan \angle APN = \frac{AN}{PA} = \frac{\text{sub-normal}}{y}$$

i. e., $$\tan \alpha = \frac{\text{sub-normal}}{y}$$

or $$\text{sub-normal} = y \times \tan \alpha = y\frac{dy}{dx}$$

To find the length of the tangent PT—

$$(PT)^2 = (PA)^2 + (AT)^2$$
$$= y^2 + y^2\left(\frac{dx}{dy}\right)^2$$
$$= y^2\left[1 + \left(\frac{dx}{dy}\right)^2\right]$$

and $$PT = y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

In like manner— $$PN = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

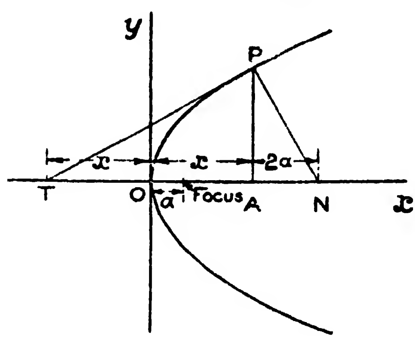


Fig. 12.

Example 12.—Find the lengths of the sub-tangent and the sub-normal of the parabola $y^2 = 4ax$ (Fig. 12).

$$y^2 = 4ax \quad \text{and} \quad y = 2\sqrt{a} \cdot x^{\frac{1}{2}}$$

then $$\frac{dy}{dx} = 2\sqrt{a} \cdot \frac{1}{2}x^{-\frac{1}{2}} \quad \text{or} \quad \frac{\sqrt{a}}{\sqrt{x}}$$

Then length of sub-tangent $= y\dfrac{dx}{dy}$

$$= y \times \frac{\sqrt{x}}{\sqrt{a}} = 2\sqrt{a}\sqrt{x} \times \frac{\sqrt{x}}{\sqrt{a}}$$
$$= \underline{2x}$$

This result illustrates an important property of the parabola and one useful in the drawing of tangents. For AT $= 2x = 2 \times$ AO, and hence to draw the tangent at any point P, drop PA perpendicular to the axis, set off OT = OA and join TP.

The length of the sub-normal AN $= y\dfrac{dy}{dx}$

$$= y \times \frac{\sqrt{a}}{\sqrt{x}} = \frac{2\sqrt{a}\sqrt{x}\sqrt{a}}{\sqrt{x}}$$

$$= \underline{2a.}$$

i. e., the length of the sub-normal is independent of the position of P, provided that the sub-normal is measured on the axis of the parabola.

Example 13.—Find the lengths of the sub-tangent and the sub-normal of the parabola— $y = 15x^2 - 2x - 9$

when $x = -2$ and also when $x = 3$.

The axis of this parabola is vertical, and consequently the sub-normal, which is measured along the x axis when given by the value of $y\dfrac{dy}{dx}$, is not constant.

Now— $y = 15x^2 - 2x - 9$

and $\dfrac{dy}{dx} = 30x - 2.$

Hence sub-tangent— $= y\dfrac{dx}{dy} = \dfrac{15x^2 - 2x - 9}{30x - 2}$

and sub-normal $= y\dfrac{dy}{dx} = (15x^2 - 2x - 9) \times (30x - 2).$

Thus when $x = -2$

$$\text{sub-tangent} = \frac{60 + 4 - 9}{-60 - 2} = -\frac{55}{62} \text{ units.}$$

$$\underline{\text{sub-normal} = (60 + 4 - 9)(-62) = -3410 \text{ units.}}$$

When $x = 3$

$$\text{sub-tangent} = \left(\frac{135 - 6 - 9}{88}\right) = \frac{120}{88} = \frac{15}{11} \text{ units.}$$

$$\underline{\text{sub-normal} = 120 \times 88 = 10560 \text{ units.}}$$

Example 14.—A shaft 24 ft. long between the bearings weighs 2 cwt. per foot run, and supports a flywheel which weighs $3\frac{1}{2}$ tons at a distance of 3 ft. from the right-hand bearing. Find at what point the maximum bending moment occurs and state the maximum bending moment.

Regarding the shaft as a simply supported beam AB (see Fig. 13), we may draw the bending moment diagrams for the respective systems

of loading, viz., ADB for the distributed load, being the weight of the shaft, and ACB for the concentrated load.

The total distributed load is wl, *i. e.*, $24\times\cdot1 = 2\cdot4$ tons, giving equal reactions of $1\cdot2$ tons at A and B; and the bending moment diagram is a parabola with vertex at D, the maximum ordinate DF being $\frac{wl^2}{8}$, *i. e.*, $\frac{\cdot1\times24^2}{8}$ or $7\cdot2$ tons ft. If for convenience in the later working the axes of x and y are as shown in the figure, the equation to this parabola is $y^2 = 4ax$; or taking the value of y as FB and that of x as DF, $12^2 = 4a\times7\cdot2$, from which $4a = 20$ and $y^2 = 20x$.

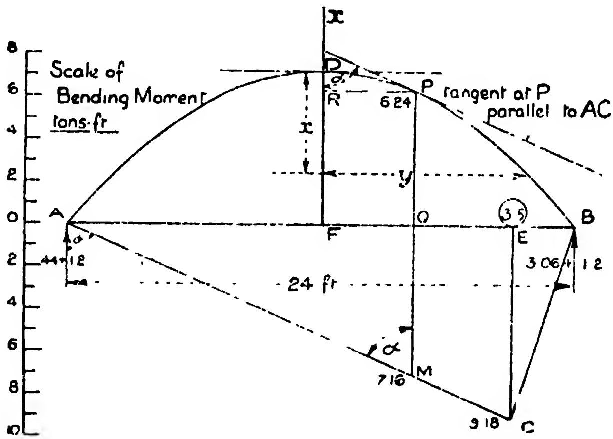


FIG. 13.

The load of $3\cdot5$ tons produces reactions of $\frac{21}{24}\times3\cdot5$ tons at B, and $\frac{3}{24}\times3\cdot5$ tons at A, *i. e.*, $R_B = 3\cdot06$ and $R_A = \cdot44$ tons: thus the bending moment at E is $3\cdot06\times3 = 9\cdot18$ tons ft.

Since the total bending moment is obtained by adding the ordinates of the diagram ADB to the corresponding ordinates of ACB, the maximum bending moment will be determined when the tangent to the parabola is parallel to AC, and the position satisfying this condition can readily be found by differentiation Thus—

The equation of the curve ADB is $y^2 = 20x$ or $y = 4\cdot47x^{\frac{1}{2}}$, and the slope of the curve is given by the value of $\frac{dy}{dx}$.

Now if $y = 4\cdot47x^{\frac{1}{2}}$, $\qquad \frac{dy}{dx} = 4\cdot47\times\frac{1}{2}x^{-\frac{1}{2}}$

i. e., $\qquad \tan\alpha = \frac{4\cdot47}{2x^{\frac{1}{2}}}$

Referring to the figure ACB, $\tan\alpha = \frac{AQ}{QM} = \frac{21}{9\cdot18}$

and thus— $$\frac{4\cdot47}{2x^{\frac{1}{2}}} = \frac{21}{9\cdot18}$$

or $$x^{\frac{1}{2}} = \frac{4\cdot47\times9\cdot18}{42}$$

i. e., $$(DR)^{\frac{1}{2}} = \frac{4\cdot47\times9\cdot18}{42}.$$

Again, $(PR)^2 = 20\times DR$, and thus $PR = \sqrt{20}\times\frac{4\cdot47\times9\cdot18}{42}$
$$= 4\cdot37 \text{ ft.}$$

Thus the maximum bending moment occurs at a distance of $12-4\cdot37$, *i. e.*, 7·63 ft. from the right-hand bearing.

To find the maximum bending moment—

$$DR = \left(\frac{4\cdot47\times9\cdot18}{42}\right)^2 = \cdot956$$

$$PQ = DF-DR = 7\cdot2-\cdot96 = 6\cdot24 \text{ tons ft.}$$

Also— $$\frac{QM}{EC} = \frac{16\cdot37}{21}\times9\cdot18 = 7\cdot16 \text{ tons ft.}$$

Hence the maximum bending moment $= 7\cdot16+6\cdot24 = \underline{13\cdot4 \text{ tons ft.}}$

Exercises 3.—On the Lengths of the Sub-tangent and Sub-normal: also Beam Problems.

1. Find the lengths of the sub-normal and sub-tangent of the curve $5y = 4x^3$ at the point for which $x = 3$.

2. If $y = \frac{gx^2}{2V^2}$, $V = 117$, and $g = 32\cdot2$, find the value of x that makes the slope of the curve 1 in 17·4.

3. A parabolic arched rib has a span of 50 ft. and a rise of 8 ft. Find the equation of the tangent of the slope of the rib. What is the slope of the tangent at the end?

4. Find the equation of the tangent to the curve $p = \frac{450}{v^{1\cdot3}}$ at the point for which $v = 5$.

In Exercises 5 to 7, y is a deflection and x a distance along the beam. Find, in each case, expressions for the Bending Moment, Shearing Force and Load. The beam is of uniform section throughout, and of span l.

5. The beam is supported at both ends, and loaded with W at the centre.

$$y = \frac{W}{2EI}\left(\frac{lx^2}{4}-\frac{x^3}{6}\right) \qquad \{x \text{ is the distance from the centre}\}.$$

6. The beam is supported at both ends, and loaded continuously with w per ft. run.

$$y = \frac{w}{2EI}\left(\frac{l^2x^2}{8}-\frac{x^4}{12}\right) \qquad \{x \text{ is the distance from the centre}\}.$$

7. A cantilever loaded with W at the free end.

$$y = \frac{W}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) \qquad \{x \text{ is the distance from the fixed end}\}.$$

8. Find the lengths of the projections on the y axis of the tangent and the normal of the parabola, $x^2 = 10b^2y + 4c$, x having the value $9a$.

9. Prove that the sub-normal (along the axis of the parabola) of the parabola $x^2 = 6y$ is constant and find the value of this constant.

10. If $EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{Px^2}{2} + C$ and $C = \frac{Pl^2}{2} - \frac{wl^3}{12}$, find the value of $\frac{d^2y}{dx^2}$

Differentiation of Exponential Functions.—The rule for differentiation already given applies only to functions involving the I.V. (usually the x) raised to some power. A method must now be found for the differentiation of exponential functions, viz., those in which the I.V. appears as exponent; such as e^{5x} or 4^x.

When concerned with the plotting of the curve $y = e^x$ (see Part I, p. 352) mention was made of the fact that, if tangents are drawn to the curve at various points, the slopes of these tangents are equal to the ordinates to the primitive curve at the points at which the tangents touch the curve. Thus the slope curve of the curve $y = e^x$ lies along the primitive, and—

$$\frac{de^x}{dx} = e^x$$

or the rate of change of the function is equal to the value of the function itself.

We may establish the result algebraically thus—

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \dots$$

Assuming that a series composed of an unlimited number of terms can be differentiated term by term and the results added to give the true derivative (this being true for all the cases with which we shall deal), then by differentiation—

$$\begin{aligned}\frac{de^x}{dx} &= 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} + \dots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \\ &= e^x.\end{aligned}$$

Another respect in which the function e^x is unique may be

noted: the sub-tangent $= y\frac{dx}{dy} = e^x \times \frac{1}{e^x} = 1$, *i. e.*, the sub-tangent is constant and equal to unity.

The curve $y = e^x$ may be usefully employed as a gauge or template for testing slopes of lines; the curve being drawn on tracing-paper and moved over the line to be tested until the curve and line have the same direction, and the ordinate of the curve being then read, any necessary change of scales being afterwards made.

The work may now be carried a stage further, so that the rule for the differentiation of e^{bx} may be found.

Referring to Part I, p. 354, we note that if the curve $y = e^x$ be plotted, then this curve represents also the equation $y = e^{bx}$ if the numbers marked along the horizontal scale used for the curve $y = e^x$ are divided by b. If, then, the slope of the construction curve, *i. e.*, that having the equation $y = e^x$, is measured, we can obtain from it the slope of the curve $y = e^{bx}$ by multiplying the slope by b, since vertical distances are unaltered, whilst horizontal distances in the case of $y = e^{bx}$ are $\frac{1}{b} \times$ corresponding horizontal distances for $y = e^x$.

Hence the slope of the curve $y = e^{bx}$ is $b \times$ slope of curve $y = e^x$

or—
$$\frac{de^{bx}}{dx} = be^{bx}$$

It should be noticed that the power of the function remains the same after differentiation, but the multiplier of the I.V. becomes after differentiation a multiplier of the function. This latter rule must be remembered throughout differentiation, viz., **any multiplier or divisor of the I.V. in the function to be differentiated must become a multiplier or divisor of the function after differentiation.**

From e^{bx} we can proceed to ae^{bx}, the result from the differentiation of which is given by—

$$\frac{dae^{bx}}{dx} = abe^{bx}$$

Example 15.—If $y = 5e^{-\frac{1}{6}x}$, find the value of $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx}5e^{-\frac{1}{6}x} = 5 \times -\frac{1}{6}e^{-\frac{1}{6}x}$$
$$= \underline{-\frac{5}{6}e^{-\frac{1}{6}x}}.$$

Referring to the last example, note that the power of e is exactly the same after differentiation as before: the factor $-\frac{1}{6}$ multiplies the I.V. in the original function, and therefore it occurs as a constant multiplier after differentiation; also the constant factor 5 remains throughout differentiation.

Example 16.—If $I = I_0 e^{-\frac{Rt}{L}}$, where I and I_0 are electrical currents, R is the resistance of a circuit, L is the self-inductance of the circuit and t is a time, find the time-rate of change of I.

This example illustrates the importance of the rate of change as compared with the change itself; for it demonstrates the fact that for an inductive circuit the change of current is often extremely rapid and consequently dangerous.

$$\text{Time rate of change of I} = \frac{dI}{dt} = \frac{d}{dt} I_0 e^{-\frac{Rt}{L}}$$

$$= I_0 \times -\frac{R}{L} e^{-\frac{Rt}{L}}$$

$$= -\frac{R}{L} I_0 e^{-\frac{Rt}{L}}$$

$$= -\frac{R}{L} I.$$

i. e., the rate of decrease of the current when the impressed E.M.F. is removed is proportional to the current at the instant the circuit is broken.

To better illustrate the example, take the case for which the current at the instant of removal of the E.M.F. is 14·5 amps., the resistance of the circuit is 6·4 ohms, and its self-inductance ·006 henry.

Then the rate of change of the current $= -\frac{6 \cdot 4}{\cdot 006} \times 14 \cdot 5 = -15470$ amps. per second, whereas the actual current is only 14·5 amps.

The expressions e^x and e^{bx} are particular forms of the more general exponential function a^x; to differentiate which we may proceed by either of two methods:—

(*a*) *Working from first principles.* In Part I, p. 470, the expansion for a^x is given, viz.—

$$a^x = 1 + x \log a + \frac{(x \log a)^2}{\underline{|2}} + \frac{(x \log a)^3}{\underline{|3}} + \ldots$$

Differentiating term by term—

$$\frac{da^x}{dx} = 0+\log a+(\log a)^2x+(\log a)^3\frac{x^2}{2}+\ldots$$

$$= \log a\{1+x\log a+\frac{(x\log a)^2}{\lfloor 2}+\frac{(x\log a)^3}{\lfloor 3}+\ldots\}$$

$$= \log a\times a^x$$

i. e., $\frac{da^x}{dx} = a^x . \log a.$

(*b*) *Assuming the result for the differentiation of* e^{bx}—

Let— $a^x = e^{bx}$

so that $a = e^b$, and therefore $\log_e a = b.$

Then— $\frac{d}{dx}a^x = \frac{d}{dx}e^{bx} = be^{bx} = \log_e a\times e^{bx}$

$= \log_e a\times a^x$

or $a^x \log a$

thus— $\frac{da^x}{dx} = a^x . \log a.$

Example 17.—Find the value of $\frac{d4^x}{dx}$

In this case $a = 4$ and $\log_e 4 = 1{\cdot}3863.$

Hence $\frac{d4^x}{dx} = \underline{1{\cdot}3863\times 4^x.}$

Note carefully that this result cannot be simplified by combining 1·3863 with 4 and writing the result as $5{\cdot}5452^x$, which is quite incorrect. The 4 alone is raised to the x power, and 1·3863 is not raised to this power.

Example 18.—Find the value of $\frac{d^2}{ds^2}(3{\cdot}6)^s$.

Here $a = 3{\cdot}6$ and $\log 3{\cdot}6 = 1{\cdot}2809.$

Thus $\frac{d}{ds}(3{\cdot}6)^s = \log 3{\cdot}6\times(3{\cdot}6)^s = 1{\cdot}2809\times(3{\cdot}6)^s.$

Then—

$$\frac{d^2}{ds^2}(3{\cdot}6)^s = \frac{d}{ds}\left\{\frac{d}{ds}(3{\cdot}6)^s\right\} = \frac{d}{ds}\{1{\cdot}2809\times(3{\cdot}6)^s\}$$

$$= 1{\cdot}2809\times\frac{d}{ds}(3{\cdot}6)^s = 1{\cdot}2809\times 1{\cdot}2809\times(3{\cdot}6)^s$$

$$= \underline{1{\cdot}64(3{\cdot}6)^s.}$$

Example 19.—Given that $s = 4e^{5t}+7e^{-5t}$, find the value of $\frac{d^2s}{dt^2}-25s$.

$$s = 4e^{5t}+7e^{-5t}$$
$$\frac{ds}{dt} = (4\times 5e^{5t})+(7\times -5e^{-5t})$$
$$= 20e^{5t}-35e^{-5t}.$$

Again— $$\frac{d^2s}{dt^2} = 100e^{5t}+175e^{-5t}$$

$$\therefore \quad \frac{d^2s}{dt^2}-25s = 100e^{5t}+175e^{-5t}-100e^{5t}-175e^{-5t} = \underline{0.}$$

Differentiation of log x.—The rule for the differentiation of logarithmic functions can be derived either from the expansion of $\log_e (1 + x)$ into a series, or by assuming the result for the differentiation of e^x. Considering these methods in turn—

(*a*) *Working from first principles.* Let $y = \log x$, *i. e.*, $\log_e x$. Then if x be increased to become $x+\delta x$, y takes a new value $y+\delta y$, and $y+\delta y = \log (x+\delta x)$.

Now $\log (x+\delta x) = \log x\left(1+\frac{\delta x}{x}\right) = \log x+\log\left(1+\frac{\delta x}{x}\right)$,

therefore—

$$(y+\delta y)-y = \log x+\log\left(1+\frac{\delta x}{x}\right)-\log x$$

i. e., $$\delta y = \log\left(1+\frac{\delta x}{x}\right)$$

Also $\log\left(1+\frac{\delta x}{x}\right)$ can be expanded into a series of the form

$$\log (1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+ \ldots \text{ (see Part I, p. 470)}$$

so that—

$$\log\left(1+\frac{\delta x}{x}\right) = \left(\frac{\delta x}{x}\right)-\frac{1}{2}\left(\frac{\delta x}{x}\right)^2+\frac{1}{3}\left(\frac{\delta x}{x}\right)^3-\frac{1}{4}\left(\frac{\delta x}{x}\right)^4+ \ldots$$

$$\therefore \quad \delta y = \left(\frac{\delta x}{x}\right)-\frac{1}{2}\left(\frac{\delta x}{x}\right)^2+\frac{1}{3}\left(\frac{\delta x}{x}\right)^3-\frac{1}{4}\left(\frac{\delta x}{x}\right)^4+ \ldots$$

Dividing all through by δx—

$$\frac{\delta y}{\delta x} = \frac{1}{x}-\frac{\delta x}{2x^2}+\frac{(\delta x)^2}{3x^3}-\frac{(\delta x)^3}{4x^4}+ \ldots$$

By sufficiently diminishing the value of δx we may make the

second and succeeding terms as small as we please, and evidently the limiting value of the series is $\frac{1}{x}$

i. e.,
$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{1}{x}$$

Hence—
$$\underline{\frac{d \log_e x}{dx} = \frac{1}{x}}$$

(*b*) *Assuming the result*—
$$\frac{de^x}{dx} = e^x$$

Let— $y = \log_e x,$ so that $x = e^y$

and
$$\frac{dx}{dy} = \frac{de^y}{dy} = e^y.$$

Now $\frac{\delta y}{\delta x} = \frac{1}{\frac{\delta x}{\delta y}}$ and consequently by considering the limiting values of these fractions—
$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

We wish to find $\frac{dy}{dx}$ and we have already obtained an expression for $\frac{dx}{dy}$.

Hence—
$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}$$

or
$$\frac{d \log_e x}{dx} = \frac{1}{x}$$

This result can be amplified to embrace the more general form, thus—
$$\frac{d}{dx} \log_e (\mathrm{A}x + \mathrm{B}) = \frac{\mathrm{A}}{\mathrm{A}x + \mathrm{B}}$$

for, in accordance with the rule given on p. 48, the A which multiplies the I.V. in the original function must appear as a multiplier after differentiation.

All these rules apply to functions involving natural logs, but

they can be modified to meet the cases in which common logs occur; for — $\log_{10} x = \cdot 4343 \log_e x$

$$\text{and hence} \qquad \frac{d \log_{10} x}{dx} = \cdot 4343 \frac{d \log_e x}{dx} = \frac{\cdot 4343}{x}$$

$$\text{and} \qquad \frac{d}{dx} \log_{10} (Ax+B) = \frac{\cdot 4343A}{Ax+B}$$

It should be observed that in all these logarithmic functions the I.V. is raised to the first power only: if the I.V. is raised to a power higher than the first, other rules, which are given later, must be employed.

Example 20.—If $y = \log_e 7x$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx} \log_e 7x = \frac{7}{7x} = \frac{1}{x}$$

or alternatively— $\log_e 7x = \log_e 7 + \log_e x$

and thus— $$\frac{d}{dx} \log_e 7x = \frac{d}{dx} \log_e 7 + \frac{d}{dx} \log_e x$$

$$= 0 + \frac{1}{x} = \frac{1}{x}$$

Example 21.—Differentiate with regard to t the expression $\log_{10} (5t-14)$ and find the numerical value of the derivative when $t = 3\cdot 2$.

$$\frac{d}{dt} \log_{10} (5t-14) = \frac{\cdot 4343 \times 5}{5t-14} = \frac{2\cdot 1715}{5t-14}$$

When $t = 3\cdot 2$

$$\frac{d}{dt} \log_{10} (5t-14) = \frac{2\cdot 1715}{16-14} = 1\cdot 0858.$$

We may check this result approximately by taking values of t $3\cdot 19$ and $3\cdot 21$ and calculating the value of $\frac{\delta \log_{10} (5t-14)}{\delta t}$.

Thus—

When $t = 3\cdot 19$, $\log_{10}(5t-14) = \log_{10}(15\cdot 95-14) = \log_{10} 1\cdot 95 = \cdot 2900$

when $t = 3\cdot 21$, $\log_{10}(5t-14) = \log_{10}(16\cdot 05-14) = \log_{10} 2\cdot 05 = \cdot 3118$

so that—

$$\delta \log_{10} (5t-14) = \cdot 3118 - \cdot 2900 = \cdot 0218$$

while $$\delta t = 3\cdot 21 - 3\cdot 19 = \cdot 02$$

and $$\frac{\delta}{\delta t} \log_{10} (5t-14) = \frac{\cdot 0218}{\cdot 02} = 1\cdot 09.$$

Differentiation of the Hyperbolic Functions, sinh x and cosh x.—Expressing the hyperbolic functions in terms of exponential functions—

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

and
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Thus to differentiate sinh x we may differentiate $\frac{e^x - e^{-x}}{2}$.

Hence—
$$\frac{d \sinh x}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}(e^x + e^{-x})$$
$$= \cosh x$$

also
$$\frac{d}{dx} \cosh x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}(e^x - e^{-x})$$
$$= \sinh x.$$

Example 22.—Find the inclination to the horizontal of a cable weighing ½ lb. per ft. and stretched to a tension of 30 lbs. weight, at the end of its span of 50 ft.

The equation to the form taken by the cable is—

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) = c \cosh \frac{x}{c}$$

where
$$c = \frac{\text{horizontal tension}}{\text{weight per foot}} = \frac{30}{\cdot 5} = 60.$$

We require the slope of the curve when $x = 25$, this being given by the value of $\frac{dy}{dx}$ there.

$$\frac{dy}{dx} = \frac{d}{dx} c \cosh \frac{x}{60} = c \times \frac{1}{60} \sinh \frac{x}{60} = \sinh \frac{x}{60}$$

When $x = 25$
$$\frac{dy}{dx} = \sinh \frac{25}{60} = \sinh \cdot 4167 = \frac{e^{\cdot 4167} - e^{-\cdot 4167}}{2}$$
$$= \frac{1 \cdot 517 - \cdot 659}{2}$$
$$= \cdot 429.$$

This value is that of the tangent of the angle of inclination to the horizontal; which is thus $\tan^{-1} \cdot 429$ or 23° 13′.

Exercises 4.—On the Differentiation of a^x, Log x and the Hyperbolic Functions.

Differentiate with respect to x the functions in Nos. 1 to 20.

1. e^{-5x}. 2. $1\cdot 5e^{4\cdot 1x}$. 3. $\frac{27}{e^{\cdot 7x}}$. 4. $4\cdot 15^x$. 5. $8\cdot 72^{2x}$.

6. $e^{9x}+5e^{-7x}$. 7. $2x^e$. 8. 14×2^x. 9. $41e^{\cdot 25x}+\frac{19}{e^{\cdot 16x}}+e^{\cdot 416}+17$.

10. $3\cdot 14e^{5\cdot 1x}-5x^{9\cdot 43}+3\cdot 1^{2x}+b$. 11. $\frac{10e^{\cdot 4x}}{5e^{3x}}\times\frac{3e^{7\cdot 2x}}{2e^{-4\cdot 6x}}$. 12. $\log 7x$.

13. $3\log(4-5x)$. 14. $10\log_{10} 8x$. 15. $c\log(4xa+5b)$.

16. $9e^{-\cdot 2x}-\log\cdot 2x+\frac{5\cdot 71}{x^{\cdot 43}}$. 17. $\log 2x(3x-4\cdot 7)$.

18. $\log\frac{(5x+4)(3x-2)}{(7-4x)}$. 19. $(e^x)^3+4\cosh 2x-1\cdot 7\log_{10} 2\cdot 3x$.

20. $\log 3x^2+5x^{-\cdot 7}-1\cdot 8(1\cdot 8^x)+12$.

21. If $y = A_1e^{-3x}+A_2e^{-4x}$, find the value of $\frac{d^2y}{dx^2}+7\frac{dy}{dx}+12y$.

22. Find $\frac{d}{dv}\log(3-4v)$ when $v = 1\cdot 7$. Check your result approximately by taking as values of v 1·65 and 1·75.

23. Determine the value of $\frac{d}{du}71\log(18-\cdot 04u)$.

24. Write down the value of $\frac{d}{dt}\log_{10} 18t$.

25. If $T = 50e^{\cdot 2\theta}$, find the rate of change of T compared with change in θ.

26. If $I = I_0\left(1-e^{-\frac{Rt}{L}}\right)$, I and I_0 being electrical currents, R the resistance of a circuit and L its self-inductance, find the rate at which the current I is changing, t being the time.

27. Given that $v = 2\cdot 03\log_{10}(7-1\cdot 8u)$, find $\frac{dv}{du}$

28. Evaluate $\frac{d}{dx}5\cosh\frac{x}{4}$ and also $\frac{d}{dy}p\sinh\frac{y}{q}$.

29. An electromotive force E is given by—

$$E = A\cosh\sqrt{lr}\,.\,x+B\sinh\sqrt{lr}\,.\,x.$$

Find the value of $\frac{d^2E}{dx^2}$ in terms of E.

30. If $W = 144\{p_1(1+\log r)-rp_b\}$, find the value of r that makes $\frac{dW}{dr} = 0$; W being the work done in the expansion of steam from pressure p_1 through a ratio of expansion r.

31. Find the value of $\frac{dy}{dx}+ay$ if $y = \frac{b}{a}-\frac{A}{a}e^{-ax}$.

32. If $y = Ae^{2x}+Be^{3x}+Ce^{-4x}$, find the value of—

$$\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-14\frac{dy}{dx}+24y.$$

33. Evaluate $\frac{d^2V}{dx^2}-\frac{Vr_1}{r_2}$ when $V=A_1e^{\sqrt{\frac{r_1}{r_2}}.x}+A_2e^{-\sqrt{\frac{r_1}{r_2}}.x}$

34. Nernst gives the following rule connecting the pressure p of a refrigerant (such as Carbon Dioxide or Ammonia) and its absolute temperature τ—

$$p = A+B\log\tau+C\tau+\frac{D}{\tau}$$

where A, B, C and D are constants. Find an expression for $\frac{dp}{d\tau}$.

Differentiation of the Trigonometric Functions.—Before proceeding to establish the rules for the differentiation of $\sin x$ and $\cos x$, it is well to remind ourselves of two trigonometric relations which are necessary for the proofs of these rules, viz.—

(*a*) When the angle is small, its sine may be replaced by the angle itself expressed in radians, *i. e.*—

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad \text{(cf. Part I, p. 458).}$$

(*b*) $\sin A-\sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\frac{(A-B)}{2}$ (cf. Part I, p. 285).

To find $\frac{d}{dx}\sin x$ we proceed as in former cases; thus—

Let $\quad y = \sin x \quad$ and $y+\delta y = \sin(x+\delta x)$

then $\quad \delta y = y+\delta y-y = \sin(x+\delta x)-\sin x$

$$= 2\cos\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$$

Dividing through by δx—

$$\frac{\delta y}{\delta x} = \frac{2\cos\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= \frac{\cos\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

$$= \cos\left(\frac{2x+\delta x}{2}\right)\times\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$$

The limiting value of $\frac{\delta y}{\delta x}$ is $\frac{dy}{dx}$, and that of the right-hand side is $\cos x$, since $\cos\left(x+\frac{\delta x}{2}\right)$ approaches more and more nearly to $\cos x$ as δx is made smaller and smaller, and the limiting value of $\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$, or, as we might write it, $\frac{\sin\theta}{\theta}$, is 1.

Hence $$\frac{d\sin x}{dx} \text{ or } \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \cos x$$

$$\frac{d\sin x}{dx} = \cos x.$$

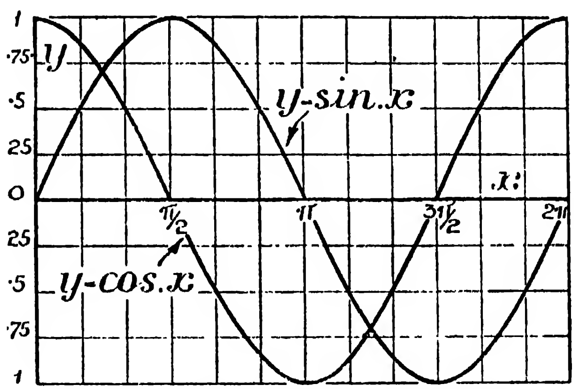


FIG. 14.—Curves of $y = \sin x$ and $y = \cos x$.

By similar reasoning the derivative of $\cos x$ may be obtained; its value being given by—

$$\frac{d\cos x}{dx} = -\sin x$$

The graphs of the sine and cosine curves assist towards the full appreciation of these results. In Fig. 14 the two curves are plotted, and it is noted that the cosine curve is simply the sine curve shifted backwards along the horizontal axis: thus the slope curve and the primitive have exactly the same shape. This condition also holds for the primitive curve $y = e^{bx}$, and so suggests that there must be some connection between these various natural functions; and further reference to this subject is made later in the book.

Much trouble is caused by the presence of the minus sign in the relation $\frac{d \cos x}{dx} = -\sin x$, it being rather difficult to remember whether the minus sign occurs when differentiating $\sin x$ or $\cos x$. A mental picture of the curves, or the curves themselves, may be used as an aid in this respect. The cosine and sine curves differ in phase by $\frac{1}{4}$ period (see Fig. 14), but are otherwise identical. Treating $y = \sin x$ as the primitive:—when x is small, $\sin x$ and x are very nearly alike, and thus the slope of the curve here is 1; as x increases from 0 to $\frac{\pi}{2}$ the slope of the curve continually diminishes until at $x = \frac{\pi}{2}$ the slope of the curve is zero. Now the ordinate of the cosine curve when $x = 0$ is unity, and it diminishes until at $x = \frac{\pi}{2}$ it is zero. From $x = \frac{\pi}{2}$ to $x = \pi$ the slope of the sine curve is negative, but increases *numerically* to -1, this being the value when $x = \pi$; and it may be observed that the ordinates of the cosine curve give these changes exactly, both as regards magnitude and sign. Thus the cosine curve is the slope curve of the sine curve.

Now regard the cosine curve as the primitive. At $x = 0$ the curve is horizontal and the slope $= 0$; from $x = 0$ to $x = \frac{\pi}{2}$ the slope increases *numerically*, but is negative, reaching its maximum negative value, viz., -1, at $x = \frac{\pi}{2}$; but the ordinates of the sine curve are all positive from $x = 0$ to $x = \frac{\pi}{2}$, so that although these ordinates give the slope of the curve as regards magnitude, they give the wrong sign. In other words, the sine curve must be folded over the axis of x to be the slope curve of the cosine curve, *i. e.*, the curve $y = -\sin x$ is the slope curve of the curve $y = \cos x$.

To summarise, we can say that the derived curve for the sine curve or for the cosine curve is the curve itself shifted back along the axis a horizontal distance equal to one-quarter of the period.

Thus we can say at once that the slope curve of the curve $y = \sin(x+b)$ is the curve $y = \cos(x+b)$, since the curve $y = \sin(x+b)$ is the simple sine curve shifted along the horizontal axis an amount given by the value of b, the amplitude and period being unaltered.

Thus— $$\frac{d}{dx}\sin(x+b) = \cos(x+b)$$

and, in like manner—

$$\frac{d}{dx}\cos(x+b) = -\sin(x+b).$$

Again, $\frac{d}{dx}\sin(5x+6) = 5\cos(5x+6)$, since 5 multiplies the I.V. in the original function.

Then in general—

$$\frac{d}{dx}\mathrm{A}\sin(\mathrm{B}x+\mathrm{C}) = \mathrm{AB}\cos(\mathrm{B}x+\mathrm{C})$$

$$\frac{d}{dx}\mathrm{A}\cos(\mathrm{B}x+\mathrm{C}) = -\mathrm{AB}\sin(\mathrm{B}x+\mathrm{C}).$$

To differentiate tan x with regard to x.

Let $y = \tan x$ then $(y+\delta y) = \tan(x+\delta x)$

$$\begin{aligned}\delta y &= y+\delta y-y=\tan(x+\delta x)-\tan x\\ &= \frac{\sin(x+\delta x)}{\cos(x+\delta x)}-\frac{\sin x}{\cos x}\\ &= \frac{\sin(x+\delta x)\cos x-\cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}\\ &= \frac{\sin[(x+\delta x)-x]}{\cos(x+\delta x)\cos x}\\ &= \frac{\sin\delta x}{\cos(x+\delta x)\cos x}\end{aligned}$$

Dividing through by δx—

$$\frac{\delta y}{\delta x} = \frac{\sin\delta x}{\delta x}\times\frac{1}{\cos(x+\delta x)\cos x}$$

Now as δx approaches zero, $\frac{\sin\delta x}{\delta x}$ approaches 1 and $(x+\delta x)$ approaches x.

Hence— $$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = 1\times\frac{1}{\cos x\cos x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$\therefore \quad \frac{d}{dx}\tan x = \sec^2 x$$

In like manner it can be proved that—

$$\frac{d \cot x}{dx} = -\operatorname{cosec}^2 x$$

$$\frac{d \sec x}{dx} = \frac{\sin x}{\cos^2 x}$$

and
$$\frac{d \operatorname{cosec} x}{dx} = -\frac{\cos x}{\sin^2 x}$$

To generalise—

$$\frac{d}{dx} A \tan (Bx+C) = AB \sec^2 (Bx+C)$$

$$\frac{d}{dx} A \cot (Bx+C) = -AB \operatorname{cosec}^2 (Bx+C)$$

$$\frac{d}{dx} A \operatorname{cosec} (Bx+C) = -\frac{AB \cos (Bx+C)}{\sin^2 (Bx+C)}$$

$$\frac{d}{dx} A \sec (Bx+C) = \frac{AB \sin (Bx+C)}{\cos^2 (Bx+C)}.$$

Example 23.—Find the slope of the curve representing the equation $s = 5{\cdot}2 \sin (40t-2{\cdot}4)$ when $t = {\cdot}07$.

The slope of the curve—

$$= \frac{ds}{dt} = \frac{d}{dt} 5{\cdot}2 \sin (40t-2{\cdot}4) = 5{\cdot}2 \times 40 \cos (40t-2{\cdot}4)$$
$$= 208 \cos (40t-2{\cdot}4).$$

Hence when—

$$t = {\cdot}07, \text{ the slope } = 208 \cos (2{\cdot}8-2{\cdot}4) = 208 \cos {\cdot}4 \text{ (radian)}$$
$$= 208 \cos 22{\cdot}9^\circ$$
$$= \underline{192.}$$

Example 24.—Differentiate, with regard to z, the function $9{\cdot}4 \cot (7-5z)$.

$$\frac{d}{dz} 9{\cdot}4 \cot (7-5z) = 9{\cdot}4 \times -5 \times - \operatorname{cosec}^2 (7-5z)$$
$$= \underline{47 \operatorname{cosec}^2 (7-5z)}.$$

Simple Harmonic Motion.—We can now make a more strict examination of simple harmonic motion. Suppose a crank of length r (see Fig. 15), starting from the position OX, rotates at a constant angular velocity ω in a right-handed direction. Let it have reached the position OA after t seconds have elapsed from the start; then the angle passed through in this interval of time $=$ AOM $= \omega t$, since the angular distance covered in 1 sec. $= \omega$ radians and the angular distance in t seconds $= \omega t$ radians.

Considering the displacement along the horizontal axis, the displacement in time $t = s = \text{OM}$

$$= \text{AO} \cos \text{AOM} = r \cos \omega t.$$

Then the velocity $= \dfrac{ds}{dt} = -r \times \omega \sin \omega t = -r\omega \sin \omega t$

and the acceleration $= \dfrac{dv}{dt} = -r\omega \times \omega \times \cos \omega t = -\omega^2 \times r \cos \omega t$

$$= -\omega^2 s$$

i. e., the acceleration is proportional to the displacement, but is directed towards the centre: thus, when the displacement *from* the centre increases, the acceleration *towards* the centre increases. When the displacement is greatest, the acceleration is greatest: *e. g.*, if the crank is in the position OX, the acceleration has its maximum value $\omega^2 r$ and is directed towards the centre, just destroying the outward velocity, which at X is zero. At O the acceleration $= -\omega^2 \times 0 = 0$, or the velocity is here a maximum.

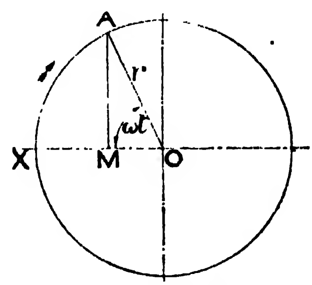


FIG. 15.

An initial lag or lead of the crank does not affect the truth of the foregoing connection between acceleration and displacement. The equation of the motion is now $s = r \cos (\omega t \pm c)$, where c is the angle of lag or lead, and the differentiation to find the velocity and the acceleration is as before.

Example 25.—If $s = 5 \sin 4t - 12 \cos 4t$, show that this is the equation of a S.H.M. and find the angular velocity.

$$s = 5 \sin 4t - 12 \cos 4t.$$

Then— $v = \dfrac{ds}{dt} = (5 \times 4 \cos 4t) - (12 \times 4 \times - \sin 4t).$

$$= 20 \cos 4t + 48 \sin 4t$$

and $a = \dfrac{dv}{dt} = (20 \times 4 \times - \sin 4t) + (48 \times 4 \cos 4t)$

$$= -80 \sin 4t + 192 \cos 4t$$
$$= -16(5 \sin 4t - 12 \cos 4t) = -16s$$

i. e., the acceleration is proportional to the displacement.

Now, in S.H.M., the acceleration $= -\omega^2 s$.

$\therefore \quad \omega^2 = 16$, *i. e.*, ω = angular velocity

= 4 radians per sec.

This last question might be treated rather differently by first expressing $5 \sin 4t - 12 \cos 4t$ in the form $M \sin (4t+c)$ (see Part I, p. 276) and then differentiating. This method indicates that a S.H.M. may be composed of two simple harmonic motions differing in phase and amplitude.

Exercises 5.—On the Differentiation of Trigonometric Functions.

Differentiate with respect to x the functions in Nos. 1 to 16.

1. $\sin (4-5{\cdot}3x)$. **2.** $3{\cdot}2 \cos 5{\cdot}1x$. **3.** $\cdot 16 \tan (3x+9)$.

4. $2{\cdot}15 \sin \left(\frac{1{\cdot}7x-5}{4}\right)$. **5.** $8 \cot 5x$.

6. $43{\cdot}15 \sec (\cdot 05 - \cdot 117x)$. **7.** $bc \cos (d-gx)$.

8. $4 \cos 5x - 7 \sin (2x-5)$. **9.** $\sin 5{\cdot}2x \cos 3{\cdot}6x$.

10. $2{\cdot}17 \cos 4{\cdot}5x \cos 1{\cdot}7x$. **11.** $9{\cdot}04 \sin (px+c) \sin (qx-c)$.

12. $5 \sin^3 x$. **13.** $\cdot 065 \cos^2 3x$.

14. $\cos^2 (7x-1{\cdot}5) + \sin^2 (7x-1{\cdot}5)$.

15. $3x^{1{\cdot}72} - 5{\cdot}14 \log (3x-4{\cdot}1) + \cdot 14 \sin (4{\cdot}31 - \cdot 195x) + 24{\cdot}93x$.

16. $7{\cdot}05 \sin \cdot 015x - \cdot 23 \cos (6{\cdot}1 - \cdot 23x) + 1{\cdot}85 \tan (4x - \cdot 07)$.

17. x, the displacement of a valve from its central position, is given approximately by $x = -1{\cdot}2 \cos \omega t - 1{\cdot}8 \sin \omega t$, where ω = angular velocity of crank shaft (making 300 r.p.m.) and t is time in seconds from dead centre position.

Find expressions for the velocity and acceleration of the valve.

18. If $s = 4{\cdot}2 \sin (2{\cdot}1 - \cdot 17t) - \cdot 315 \cos (2{\cdot}1 - \cdot 17t)$, s being a displacement and t a time, find an expression for the acceleration in terms of s. What kind of motion does this equation represent?

19. The current in a circuit is varying according to the law $I = 3{\cdot}16 \sin (2\pi ft - 3{\cdot}06)$. At what rate is the current changing when $t = \cdot 017$, the frequency f being 60?

20. If the deflected form of a strut is a sine curve, what will be the form of the bending moment curve?

21. If y = deflection of a rod at a distance x from the end, the end load applied being F—

$$y = \frac{\frac{Bl}{8} \cos \frac{\pi x}{l}}{\frac{EI\pi^2}{l^2} - F}$$

Find the value of $EI\frac{d^2y}{dx^2} + Fy + \frac{Bl}{8} \cos \frac{\pi x}{l}$; y and x being the only variables.

22. The primary E.M.F. of a certain transformer was given by the expression—

$$E = 1500 \sin pt + 100 \sin 3pt - 42 \cos pt + 28 \cos 3pt.$$

Find the rate at which the E.M.F. varied.

23. A displacement s is given by $s = \sin 12t - \frac{12}{13} \sin 13t$. Show that the acceleration $= 25 \sin 12t - 169s$.

CHAPTER III

ADDITIONAL RULES OF DIFFERENTIATION

Differentiation of a Function of a Function.—Whilst the expression $e^{\sin 4x}$ is essentially a function of x, it can also be spoken of as a function of $\sin 4x$, which in turn is a function of x; and thus it is observed that $e^{\sin 4x}$ is a function of a function of x. This fact will be seen more clearly, perhaps, if u is written in place of $\sin 4x$: thus $e^{\sin 4x} = e^u$, which is a function of u, which, again, is a function of x, since $u = \sin 4x$.

To differentiate a function of a function the following rule is employed—

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

and this rule is easily proved.

Let y be a function of u, and let u be a function of x: then y is a function of a function of x. Now increase x by a small amount δx; then since u depends on x, it takes a new value $u + \delta u$, and also the new value of y becomes $y + \delta y$. Since these changes are measurable quantities, although small, the ordinary rules of arithmetic can be applied, so that—

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

When δx approaches zero these fractions approach the limiting values $\frac{dy}{dx}$, $\frac{dy}{du}$ and $\frac{du}{dx}$ respectively; and thus in the limit—

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

In like manner, if y is a function of u, u a function of w, and w a function of x, it can be proved that—

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dw} \times \frac{dw}{dx}$$

It will be observed that on the right-hand side of the equation we have dy as the first numerator and dx as the last denominator (these giving in conjunction the left-hand side of the equation); and we may regard the other numerators and denominators as neutralising one another. The simple arithmetic analogy may help to impress the rule upon the memory: thus—

$$\frac{4}{5}=\frac{4}{51}\times\frac{51}{7\cdot6}\times\frac{7\cdot6}{5}.$$

Example 1.—If $y = e^{\sin 4x}$, find the value of $\frac{dy}{dx}$.

Let $u = \sin 4x$, so that $\frac{du}{dx} = 4\cos 4x$ and $y = e^u$.

Since y is now a function of u, we can differentiate it with regard to u, whereas it is impossible to differentiate with regard to x directly.

$$y = e^u \text{ and } \qquad \frac{dy}{du} = \frac{de^u}{du} = e^u = e^{\sin 4x}.$$

Then, since
$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$
$$\frac{dy}{dx} = e^{\sin 4x}\times 4\cos 4x$$
$$= \underline{4\cos 4x\times e^{\sin 4x}}.$$

Example 2.—Find the value of $\frac{d}{dx}\log(\cos 2x)^3$.

Let $v = (\cos 2x)^3$ and $u = \cos 2x$; and thus $y = \log_e v$ and $v = u^3$

Then
$$\frac{dy}{dx} = \frac{dy}{dv}\times\frac{dv}{du}\times\frac{du}{dx}$$
$$= \frac{d\log v}{dv}\times\frac{du^3}{du}\times\frac{d\cos 2x}{dx}$$
$$= \frac{1}{v}\times 3u^2\times -2\sin 2x$$
$$= \frac{-6\sin 2x\times(\cos 2x)^2}{(\cos 2x)^3} = \frac{-6\sin 2x}{\cos 2x}$$
$$= \underline{-6\tan 2x}.$$

$$u = \cos 2x \qquad \frac{du}{dx} = -2\sin 2x$$
$$v = u^3 \qquad \frac{dv}{du} = 3u^2$$

Example 3.—The radius of a sphere is being decreased at the rate of ·02 in. per min. At what rate is (*a*) the surface, (*b*) the weight, varying, when the radius is 15 ins. and the material weighs ·3 lb. per cu. in.?

If r = radius, then $\frac{dr}{dt}$ = rate of change of the radius, and is in this case equal to $-\cdot02$.

(*a*) The surface $= 4\pi r^2$, and thus the rate of change of surface

$$
\begin{aligned}
&= \frac{dS}{dt} \\
&= \frac{d}{dt}(4\pi r^2) \\
&= 4\pi \cdot \frac{dr^2}{dt} \\
&= 4\pi \cdot \frac{dr^2}{dr} \times \frac{dr}{dt} \\
&= 4\pi \times 2r \times \frac{dr}{dt} \\
&= 8\pi r \times -\cdot 02.
\end{aligned}
$$

Hence when $r = 15$, $\frac{dS}{dt} = 8\pi \times 15 \times -\cdot 02 = \underline{-7\cdot 53}$, *i. e.*, the surface is being diminished at the rate of 7·53 sq. ins. per min.

(*b*) The volume $= \frac{4}{3}\pi r^3$

so that the rate of change of volume $= \frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$

and the rate of change of the weight $= \frac{dW}{dt} = \frac{d}{dt}\left(\frac{4}{3} \times \cdot 3\pi r^3\right)$

$$
\begin{aligned}
\frac{dW}{dt} = \frac{d}{dt}(\cdot 4\pi r^3) = \cdot 4\pi \cdot \frac{dr^3}{dt} &= \cdot 4\pi \times \frac{dr^3}{dr} \times \frac{dr}{dt} \\
&= \cdot 4\pi \times 3r^2 \times \frac{dr}{dt} \\
&= \cdot 4\pi \times 3r^2 \times -\cdot 02.
\end{aligned}
$$

When $r = 15$, $\frac{dW}{dt} = \cdot 4\pi \times 3 \times 225 \times -\cdot 02 = -16\cdot 93$

or the weight is decreasing at the rate of 16·93 lbs. per min.

Example 4.—Find expressions for the velocity and acceleration of the piston of a horizontal steam engine when the crank makes n revolutions per second.

In each turn the angle swept out $= 2\pi$ radians.

Hence in 1 second $2\pi n$ radians are swept out, *i. e.*, the angular velocity $= 2\pi n$; and this is the rate of change of angle, so that $\frac{d\theta}{dt} = 2\pi n$.

From Fig. 16 $\quad CD = l \sin \alpha$

and $\quad CD = r \sin \theta$

Thus $\quad l \sin \alpha = r \sin \theta$

or $\quad \sin \theta = \frac{l}{r} \sin \alpha$, and $\sin \alpha = \frac{r}{l} \sin \theta$.

Again, $\cos \alpha = \sqrt{1-\sin^2 \alpha}$

$$= \sqrt{1-\frac{r^2}{l^2}\sin^2\theta} = \left(1-\frac{r^2}{l^2}\sin^2\theta\right)^{\frac{1}{2}}.$$

If the connecting rod is long compared with the crank, $\frac{r}{l}$ is small and $\frac{r^2}{l^2}$ still smaller, so that our method of approximation can be applied to the expansion of the bracket, *i. e.*—

$$\cos \alpha = 1-\frac{1}{2}\frac{r^2}{l^2}\sin^2\theta, \text{ very nearly.}$$

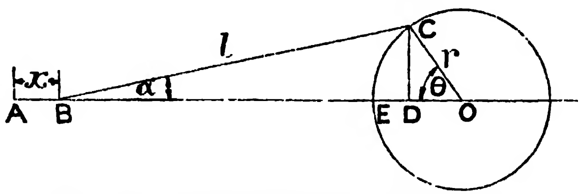


FIG. 16.—Velocity and Acceleration of Piston.

Let AB = displacement of the piston from its in-dead-centre position

$$= x = AE + OE - BO = l + r - BD - DO$$
$$= l + r - l\cos\alpha - r\cos\theta$$
$$= l + r - l\left(1-\frac{r^2}{2l^2}\sin^2\theta\right) - r\cos\theta$$
$$= r + \frac{r^2}{2l}\sin^2\theta - r\cos\theta$$
$$= r + \frac{r^2}{2l}\left(\frac{1-\cos 2\theta}{2}\right) - r\cos\theta$$
$$= r + \frac{r^2}{4l} - \frac{r^2\cos 2\theta}{4l} - r\cos\theta.$$

Now the velocity of the piston $= \frac{dx}{dt} = \frac{d}{dt}\left\{r+\frac{r^2}{4l}-\frac{r^2\cos 2\theta}{4l}-r\cos\theta\right\}$

We cannot differentiate this expression directly, so we write—

$$\frac{dx}{dt} = \frac{dx}{d\theta}\times\frac{d\theta}{dt}.$$

Hence $\frac{dx}{dt} = \frac{d}{d\theta}\left\{r+\frac{r^2}{4l}-\frac{r^2\cos 2\theta}{4l}-r\cos\theta\right\}\times\frac{d\theta}{dt}$

$$= \left\{0+0-\left(\frac{r^2}{4l}\times -2\sin 2\theta\right)-(r\times -\sin\theta)\right\}\times 2\pi n$$
$$= 2\pi n r\left\{\frac{r\sin 2\theta}{2l}+\sin\theta\right\}$$

or if $\frac{l}{r} = m$

$$v = \frac{dx}{dt} = \underline{2\pi n r\left\{\frac{\sin 2\theta}{2m}+\sin\theta\right\}}$$

Also the acceleration $= \frac{dv}{dt} = \frac{dv}{d\theta} \times \frac{d\theta}{dt} = \frac{d}{d\theta} \cdot 2\pi nr \left\{ \frac{\sin 2\theta}{2m} + \sin \theta \right\} \times \frac{d\theta}{dt}$

$$= 2\pi nr \left\{ \frac{\cos 2\theta}{m} + \cos \theta \right\} \times 2\pi n$$

$$= \underline{4\pi^2 n^2 r \left\{ \cos \theta + \frac{\cos 2\theta}{m} \right\}}$$

Example 5.—Water is flowing into a large tank at the rate of 200 gallons per min. The reservoir is in the form of a frustum of a pyramid, the length of the top being 40 ft. and width 28 ft., and the corresponding dimensions of the base being 20 ft. and 14 ft.; the depth is 12 ft. (see Fig. 17). At what rate is the level of the water rising when the depth of water is 4 ft.?

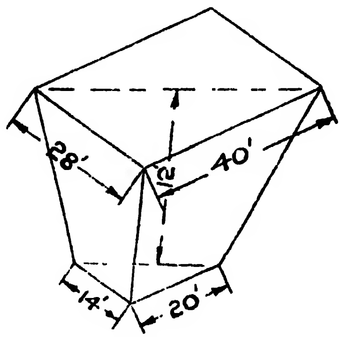


FIG. 17.

In 12 ft. the length decreases by 20 ft., and therefore in 8 ft. the length decreases by $\frac{20 \times 8}{12}$, *i. e.*, $13\frac{1}{3}$ ft., so that the length when the water is 4 ft. deep is $40 - 13\frac{1}{3} = 26\frac{2}{3}$ ft.

Similarly, the breadth $= 28 - (\frac{2}{3} \times 14) = 18\frac{2}{3}$ ft.

i. e., the area of surface $= 26\frac{2}{3} \times 18\frac{2}{3} = 498$ sq. ft.

200 gals. per min. $= \frac{200}{6 \cdot 24}$ cu. ft. per min.

$= 32 \cdot 1$ cu. ft. per min.

i. e., the rate of change of volume $= \frac{dV}{dt} = 32 \cdot 1$.

Now $\frac{dV}{dt} = \frac{d}{dt} \cdot Ah$, where A = area of surface

and h = depth of water,

$= A\frac{dh}{dt}$ $\left\{ \begin{array}{l} \text{since for the short interval of time considered the} \\ \text{area of the surface may be considered constant.} \end{array} \right\}$

Hence the rate of change of level $= \frac{dh}{dt} = \frac{dV}{dt} \times \frac{1}{A}$

$= \frac{32 \cdot 1 \times 1}{498} = \cdot 0644$ ft. per min.

$=$ <u>$\cdot 773$ in. per min.</u>

Example 6.—If a curve of velocity be plotted to a base of space, prove that the sub-normal of this curve represents the acceleration.

The sub-normal of a curve $= y\frac{dy}{dx}$ (see p. 43).

In this case, since v is plotted along the vertical axis and s along the horizontal axis—

$$
\begin{aligned}
\text{the sub-normal} &= v\frac{dv}{ds} \\
&= v \cdot \frac{dv}{dt} \times \frac{dt}{ds} \\
&= v \times a \times \frac{1}{v} \\
&= a
\end{aligned}
$$

$$
\left\{\begin{array}{ll}
\text{for} & \frac{dv}{dt} = \text{rate of change of velocity} = a \\
\text{and} & \frac{ds}{dt} = \text{rate of change of position} = v
\end{array}\right\}.
$$

As a further example of this rule, consider the case of motion due to gravity; in this instance $v^2 = 2gs$, *i. e.*, the velocity space curve is a parabola. Hence we know that the sub-normal must be a constant, *i. e.*, the acceleration must be constant.

$$\text{The sub-normal} = v\frac{dv}{ds}$$

$$\text{Now} \qquad \frac{dv^2}{ds} = \frac{d}{ds} \cdot 2gs = 2g \cdot \frac{ds}{ds} = 2g$$

$$\text{but} \qquad \frac{dv^2}{ds} = \frac{dv^2}{dv} \cdot \frac{dv}{ds} = 2v\frac{dv}{ds}$$

$$2v\frac{dv}{ds} = 2g$$

$$\text{or} \qquad v\frac{dv}{ds} = g$$

i. e., the sub-normal or the acceleration $= g$.

Exercises 6.—On the Differentiation of a Function of a Function.

Find 1. $\frac{d}{dx} \cdot e^{\sin 2x}$. 2. $\frac{d}{dv} \log v^3$. 3. $\frac{u}{dt} 2\cos^2 t$.

4. $\frac{d}{dx} 8 \sin x^3$. 5. $\frac{d}{dx}\, 3{\cdot}14 \tan (5x^2+7x-2)$. 6. $\frac{d}{dx} a^{\sin 3x}$.

7. $\frac{d}{dx} e^{e^{1\cdot 85}}$. 8. $\frac{d}{dx} \log_{10} (3+7x-9x^2)$. 9. $\frac{d}{ds} \cos (\log s^5)$.

and 10. $\frac{d}{dx} \log \tan \frac{x}{2}$.

11. In the consideration of the theory of Hooke's coupling it is required to find an expression for $\frac{\omega_B}{\omega_A}$, *i. e.*, a ratio of angular velocities

If $\omega_B = \frac{d\phi}{dt}$, $\omega_A = \frac{d\theta}{dt}$ and $\tan\phi = \frac{\tan\theta}{\cos\alpha}$, find an expression for $\frac{\omega_B}{\omega_A}$ in terms of the ratios of θ, ϕ and α.

12. Find an expression for the slope of the cycloid at any point. The equation of the cycloid is

$$x = a(\theta + \sin\theta)$$
$$y = a(1 + \cos\theta)$$

the co-ordinates x and y being measured as indicated in Fig. 18.

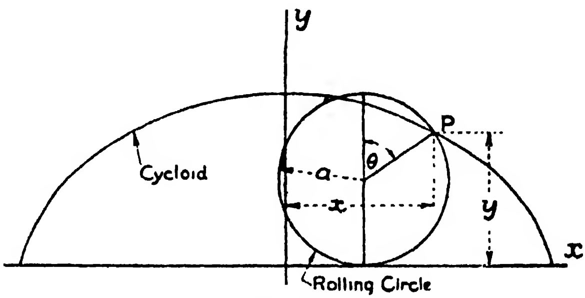


FIG. 18.

13. Assuming that the loss of head due to turbulent flow of water in a pipe is expressed by $h = C(AV^2 + BV^{\frac{3}{2}})$, where V = mean velocity of flow in ft. per sec.; show that the slope of the curve in which log h and log V are plotted with rectangular co-ordinates is given by—

$$\frac{d \log h}{d \log V} = \frac{2V^{\frac{1}{2}} + \frac{3B}{2A}}{V^{\frac{1}{2}} + \frac{B}{A}}$$

14. If $3x^2 + 8xy + 5y^2 = 1$

show that $$\frac{d^2y}{dx^2} = \frac{1}{(4x+5y)^3}.$$

15. A vessel in the form of a right circular cone whose height is 7 ft. and diameter of its base 6 ft., placed with its axis vertical and vertex downwards, is being filled with water at the rate of 10 cu. ft. per min.; find the velocity with which the surface is rising (*a*) when the depth of the water is 4 ft. and (*b*) when 60 cu. ft. have been poured in.

16. If $\frac{1}{1-E} = (r)^{\frac{R}{K}}$, prove that $\frac{dE}{dK} = -\frac{R}{K^2}\left(\frac{1}{r}\right)^{\frac{R}{K}} \log r$.

17. If $x^3 - 6x^2y - 6xy^2 + y^3 =$ constant, prove that—

$$\frac{dy}{dx} = \frac{x^2 - 4xy - 2y^2}{2x^2 - y^2 + 4xy}.$$

18. A ring weight is being turned in a lathe. It is required to find the weight removed by taking a cut of depth $\frac{1}{200}''$. The material is cast iron (·26 lb. per cu. in.), the outside diameter of the ring is 3·26″ and the length is 2·5″. Find the weight removed.

Find also a general expression for the weight removed for a cut of depth $\frac{1}{200}''$ at any diameter.

19. Find the value of $\frac{d}{dt}\left\{\log \tan \frac{7t}{2}+15t^3\right\}$.

20. If $P = \frac{F^2l}{2EA}$, and $\frac{dF}{dW} = u$, find $\frac{dP}{dW}$. (This question has reference to stresses in redundant frames.)

21. Find the angle which the tangent to the ellipse $\frac{x^2}{4}+\frac{y^2}{9} = 2$ at the point $x = 2$, $y = -3$, makes with the axis of x.

22. Find the slope of the curve $4x^2+4y^2 = 25$ at the point $x = 2$, $y = -\frac{3}{2}$, giving the angle correct to the nearest minute.

23. If force can be defined as the space-rate of change of kinetic energy, and kinetic energy $= \frac{wv^2}{2g}$, prove that force $= \frac{wa}{g}$.

24. If $x = 8 \log (12t^3-74)$, find the value of $\frac{dx}{dt}$.

Differentiation of a Product of Functions of x.—It has already been seen that to differentiate the sum of a number of terms we differentiate the terms separately and add the results. We might therefore be led to suppose that the differentiation of a product might be effected by a somewhat similar plan, viz., by multiplication together of the derivatives of the separate factors. This is, however, not the correct procedure; thus—

$$\frac{d}{dx}(\log x\times x^2) \text{ does } \textit{not} \text{ equal } \frac{d \log x}{dx}\times\frac{dx^2}{dx},\ i.\,e., \frac{1}{x}\times 2x \quad \text{or} \quad 2.$$

The true rule is expressed in the following manner: If u and v are both functions of x, and $y = uv$, *i. e.*, their product—

$$\frac{dy}{dx} = \frac{d}{dx}(uv) = v\frac{du}{dx}+u\frac{dv}{dx}$$

Proof.—Let x increase by an amount δx; then since both u and v are dependent on x, u changes to a new value $u+\delta u$ and v becomes $v + \delta v$.

Now $y = uv$, and hence the new value of y, which can be written $y+\delta y$, is given by—

$$y+\delta y = (u+\delta u)(v+\delta v)$$

but $$y = uv$$

whence by subtraction—

$$\begin{aligned}\delta y = y+\delta y-y &= (u+\delta u)(v+\delta v)-uv\\ &= uv+u\delta v+v\delta u+\delta u\,.\,\delta v-uv\\ &= u\delta v+v\delta u+\delta u\,.\,\delta v.\end{aligned}$$

Dividing through by δx—

$$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x}+v\frac{\delta u}{\delta x}+\delta u\frac{\delta v}{\delta x}$$

As δx is decreased without limit, $\frac{\delta y}{\delta x}$, $\frac{\delta v}{\delta x}$ and $\frac{\delta u}{\delta x}$ approach the values $\frac{dy}{dx}$, $\frac{dv}{dx}$ and $\frac{du}{dx}$ respectively, and the term $\delta u\,.\,\frac{\delta v}{\delta x}$ becomes negligible; so that in the limit—

$$\frac{dy}{dx} = v\frac{du}{dx}+u\frac{dv}{dx}$$

The rule may be extended to apply to the case of a product of more than two functions of x. Thus if u, v and w are each functions of x—

$$\begin{aligned}\frac{d(uvw)}{dx} &= \frac{d(w\text{V})}{dx}, \text{ where V is written for } uv\\ &= w\frac{d\text{V}}{dx}+\text{V}\frac{dw}{dx}\\ &= w\frac{d(uv)}{dx}+uv\frac{dw}{dx}\\ &= w\left(v\frac{du}{dx}+u\frac{dv}{dx}\right)+uv\frac{dw}{dx}\end{aligned}$$

and thus—

$$\frac{d(uvw)}{dx} = wv\frac{du}{dx}+wu\frac{dv}{dx}+uv\frac{dw}{dx}$$

Example 7.—Find $\frac{dy}{dx}$ when $y = x^2 . \log x$.

Let— $u = x^2$ so that $\frac{du}{dx} = 2x$

and let $v = \log x$ so that $\frac{dv}{dx} = \frac{1}{x}$.

Then $\frac{d\,.\,uv}{dx} = v\frac{du}{dx}+u\frac{dv}{dx} = (\log x \times 2x)+\left(x^2\,.\,\frac{1}{x}\right)$

$= \underline{x(1+2\log x).}$

Example 8.—Find the value of $\frac{d}{dt}[5e^{-7t} \cdot \sin(6t-4)]$

Let $u = 5e^{-7t}$ so that $\frac{du}{dt} = 5 \times -7e^{-7t} = -35e^{-7t}$

and let $v = \sin(6t-4)$ so that $\frac{dv}{dt} = 6\cos(6t-4)$.

Then—
$$\frac{d \cdot uv}{dt} = v \cdot \frac{du}{dt} + u \cdot \frac{dv}{dt}$$
$$= [\sin(6t-4) \times -35e^{-7t}] + [5e^{-7t} \times 6\cos(6t-4)]$$
$$= \underline{5e^{-7t}[6\cos(6t-4) - 7\sin(6t-4)]}.$$

Example 9.—If $2q + \frac{1}{x}\frac{d}{dx}(px^2) = 0$, show that $2q = -2p - x\frac{dp}{dx}$ p being a function of x. This example has reference to thick spherical shells.

If p is a function of x, px^2 is of the form uv, where $u = p$ and $v = x^2$.

Hence—
$$\frac{d}{dx} \cdot px^2 = x^2\frac{dp}{dx} + p\frac{dx^2}{dx} = x^2\frac{dp}{dx} + 2xp.$$

Hence—
$$2q + \frac{1}{x}\frac{d}{dx} \cdot px^2 = 2q + x\frac{dp}{dx} + 2p$$

i. e.,
$$0 = 2q + x\frac{dp}{dx} + 2p$$

or
$$\underline{2q = -2p - x\frac{dp}{dx}}.$$

Example 10.—Find the value of $\frac{d}{dx}9x^4 \sin(3x-7)\log(1-5x)$.

Let $u = x^4$, $v = \sin(3x-7)$ and $w = \log(1-5x)$

then $\frac{du}{dx} = 4x^3$, $\frac{dv}{dx} = 3\cos(3x-7)$ and $\frac{dw}{dx} = \frac{-5}{1-5x} = \frac{5}{5x-1}$

$$\frac{d}{dx}9x^4 \sin(3x-7)\log(1-5x) = 9\frac{d}{dx}(uvw)$$
$$= 9\left[wv\frac{du}{dx} + wu\frac{dv}{dx} + uv\frac{dw}{dx}\right]$$
$$= 9\Big[\log(1-5x)\sin(3x-7)4x^3 + \{\log(1-5x)x^4 \times 3\cos(3x-7)\} + x^4\sin(3x-7)\frac{5}{5x-1}\Big]$$
$$= 9x^3\Big[4\sin(3x-7)\log(1-5x) + 3x\cos(31x-7)\log(1-5x) + \frac{5x\sin(3x-7)}{5x-1}\Big].$$

Exercises 7.—On the Differentiation of a Product.

Differentiate, with respect to x, the functions in Nos. 1 to 12.

1. $x^3 \sin 3x$.
2. $\log 5x \times 2x^{3\cdot 4}$.
3. $e^{9x} \log_{10} 9x$.
4. $4x^{-5}.\tan (3\cdot 1 - 2\cdot 07x)$.
5. $\cos 3\cdot 2x \cos (1\cdot 95x+4)$.
6. $\cos (5-3x) \tan 2x$.
7. $8x^{1\cdot 6} \cos (3+8x)+16x^{1\cdot 6}$.
8. $9 \log x^3 . 5^{2x}$
9. $e^{x \log x}$.
10. $\frac{5}{4}e^{4x}\left\{x^2-\frac{x}{2}+\frac{1}{8}\right\}$.
11. $6e^{3x+2}(5x+2)^3$.
12. $7\cdot 2 \tan \frac{x}{8} \log x^7$.
13. If $y = Ae^{2x} \cos \left(\frac{2x}{3}+B\right)$, find the value of—

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+\frac{85}{9}y.$$

14. Find the value of $\frac{d}{dt}e^{-5t} \cosh (-5t)$.
15. $y = (A+Bx)e^{-4x}$; find the value of $\frac{d^2y}{dx^2}+8\frac{dy}{dx}+16y$.
16. If $V = 250 \sin (7t - \cdot 116)$, $A = 7\cdot 2 \sin 7t$ and $W = VA$, find the value of $\frac{dW}{dt}$.
17. Differentiate with respect to t the function $15t^2 \sin (4-\cdot 8t)$.
18. Find the value of $\frac{d}{dt}(4t^{3\cdot 7} \cos 3t)$.

Differentiation of a Quotient.—If u and v are both functions of x, and $y = \frac{u}{v}$, then—

$$\frac{dy}{dx} = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Proof.

(*a*) *From first principles.*—Let $y = \frac{u}{v}$: then a change δx in x causes changes of δy in y, δu in u, and δv in v, so that the new value of $y = y+\delta y = \frac{u+\delta u}{v+\delta v}$.

Then— $$\delta y = y+\delta y-y = \frac{u+\delta u}{v+\delta v}-\frac{u}{v} = \frac{uv+v\delta u-uv-u\delta v}{v(v+\delta v)}$$
$$= \frac{v\delta u-u\delta v}{v(v+\delta v)}$$

and, dividing through by δx—

$$\frac{\delta y}{\delta x} = \frac{1}{v(v+\delta v)}\left\{v \cdot \frac{\delta u}{\delta x}-u \cdot \frac{\delta v}{\delta x}\right\}.$$

When δx becomes very small, $\frac{\delta y}{\delta x}$, $\frac{\delta u}{\delta x}$ and $\frac{\delta v}{\delta x}$ approach the values $\frac{dy}{dx}$, $\frac{du}{dx}$ and $\frac{dv}{dx}$ respectively, whilst $v+\delta v$ becomes indistinguishable from v.

Hence in the limit $\frac{dy}{dx} = \frac{1}{v\times v}\left(v\cdot\frac{du}{dx}-u\cdot\frac{dv}{dx}\right)$

$$= \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

(b) *Using the rules for a product and a function of a function.*

$$y = \frac{u}{v} = uv^{-1}$$

Then—

$$\frac{dy}{dx} = \frac{d}{dx}\cdot(uv^{-1}) = v^{-1}\frac{du}{dx}+u\cdot\frac{dv^{-1}}{dx}$$

$$= \left(\frac{1}{v}\frac{du}{dx}\right)+\left(u\cdot\frac{dv^{-1}}{dv}\times\frac{dv}{dx}\right)$$

$$= \left(\frac{1}{v}\cdot\frac{du}{dx}\right)+\left(u\times-1v^{-2}\times\frac{dv}{dx}\right)$$

$$= \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Example 11.—Differentiate, with regard to s, the expression—

$$\frac{4s^3+7s}{5\cos(3s+4)}.$$

Let- $u = 4s^3+7s$, then $\frac{du}{ds} = 12s^2+7$,

and let $v = 5\cos(3s+4)$, then $\frac{dv}{ds} = -15\sin(3s+4)$.

$$\text{Then } \frac{d}{ds}\cdot\left(\frac{u}{v}\right) = \frac{v\frac{du}{ds}-u\frac{dv}{ds}}{v^2}$$

$$= \frac{[5\cos(3s+4)\times(12s^2+7)]-[(4s^3+7s)\times-15\sin(3s+4)]}{25\cos^2(3s+4)}$$

$$= \frac{(12s^2+7)[\cos(3s+4)]+(12s^3+21s)[\sin(3s+4)]}{5\cos^2(3s+4)}.$$

Example 12.—If $y = 9^{4x} \times \frac{1}{\log 7x}$, find the value of $\frac{dy}{dx}$.

Let $\quad u = 9^{4x}$, then $\frac{du}{dx} = 4 \times 9^{4x} \log_e 9 = 4 \times 2{\cdot}1972 \times 9^{4x}$

$$= 8{\cdot}789 \times 9^{4x}$$

and let $\quad v = \log 7x$, then $\frac{dv}{dx} = \frac{7}{7x} = \frac{1}{x}$.

Hence
$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(\log 7x \times 8{\cdot}79 \times 9^{4x}) - \left(9^{4x} \times \frac{1}{x}\right)}{(\log 7x)^2}$$

$$= \frac{9^{4x}\{(8{\cdot}79x \times \log 7x) - 1\}}{x\,(\log 7x)^2}$$

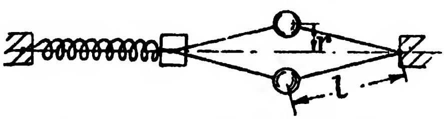


FIG. 19.—Spring loaded Governor.

Example 13.—For a spring loaded governor (see Fig. 19)—

$$\omega^2 = \frac{g\{T + 2Q(l - \sqrt{l^2 - r^2})\}}{W\sqrt{l^2 - r^2}}$$

where Q = force to elongate the spring 1 unit, T = tension in spring, W = weight of 1 ball, ω = angular velocity, r = radius of path of balls, l = length of each of the 4 arms.

If W = 3, $g = 32{\cdot}2$ and $\frac{d\omega}{dr} = 80$ when $\omega = 26$, $r = {\cdot}25$ and $l = 1$, find T and Q.

As there are two unknowns, we must form two equations. By simple substitution—

$$(26)^2 = \frac{32{\cdot}2\{T + 2Q(1 - \sqrt{1 - {\cdot}0625})\}}{3\sqrt{1 - {\cdot}0625}} \qquad \sqrt{1 - {\cdot}0625} = {\cdot}968$$

whence $\quad T + {\cdot}064Q = 60{\cdot}96 \quad \ldots \quad (1)$

We are told that $\frac{d\omega}{dr}$ must equal 80.

Now
$$\frac{d\omega^2}{dr} = \frac{d\omega^2}{d\omega} \times \frac{d\omega}{dr} = 2\omega\frac{d\omega}{dr} \quad \ldots \quad (2)$$

Also
$$\frac{d}{dr}\frac{g\{T + 2Q(l - \sqrt{l^2 - r^2})\}}{W\sqrt{l^2 - r^2}} = \frac{d}{dr}\left(\frac{u}{v}\right)$$

where $\quad u = g\{T + 2Q(l - \sqrt{l^2 - r^2})\}$

and $\quad v = W\sqrt{l^2 - r^2}$

Thus to determine $\frac{du}{dr}$ and $\frac{dv}{dr}$ it is first necessary to find the value of $\frac{d\sqrt{l^2-r^2}}{dr}$: to do this let $l^2-r^2 = y$

$$\text{so that} \quad \frac{dy}{dr} = -2r$$

then
$$\frac{d}{dr}\sqrt{l^2-r^2} = \frac{dy^{\frac{1}{2}}}{dr} = \frac{dy^{\frac{1}{2}}}{dy} \times \frac{dy}{dr}$$
$$= \tfrac{1}{2}y^{-\frac{1}{2}} \times -2r$$
$$= -\frac{r}{\sqrt{l^2-r^2}}$$

Thus
$$\frac{du}{dr} = g\left\{0+0+\frac{2Qr}{\sqrt{l^2-r^2}}\right\} = \frac{2gQr}{\sqrt{l^2-r^2}}$$

and
$$\frac{dv}{dr} = -\frac{Wr}{\sqrt{l^2-r^2}}.$$

Then
$$\frac{d}{dr}\frac{u}{v} = \frac{v\frac{du}{dr} - u\frac{dv}{dr}}{v^2}$$
$$= \frac{W\sqrt{l^2-r^2} \times \frac{2gQr}{\sqrt{l^2-r^2}} + \frac{g\{T+2Q(l-\sqrt{l^2-r^2})\}Wr}{\sqrt{l^2-r^2}}}{W^2(l^2-r^2)} \qquad (3)$$

Thus, differentiating both sides of the original equation with respect to r, we have from (2) and (3)—

$$2\omega\frac{d\omega}{dr} = \frac{gr}{W}\left\{\frac{2Q+\frac{1}{\sqrt{l^2-r^2}}[T+2Q(l-\sqrt{l^2-r^2})]}{l^2-r^2}\right\}$$

Substituting the numerical values—

$$2 \times 26 \times 80 = \frac{32 \cdot 2 \times \cdot 25}{3 \times \cdot 9375}\left\{2Q+\frac{1}{\cdot 968}[T+\cdot 064Q]\right\}$$

$$\frac{52 \times 80 \times 3 \times \cdot 9375 \times \cdot 968}{32 \cdot 2 \times \cdot 25} = 1 \cdot 936Q+T+\cdot 064Q = 2Q+T$$

whence $\quad 1407 = 2Q+T$

but from (1) $\quad 60 \cdot 96 = \cdot 064Q+T$

and therefore $\quad Q = 695 \cdot 3$

and $\quad T = 16 \cdot 4$

Differentiation of Inverse Trigonometric Functions.—Since inverse trigonometric functions occur frequently in the study of the Integral Calculus, it is necessary to demonstrate the rules for their differentiation; and in view of their importance in the later stages of the work, the results now to be deduced should be carefully studied.

The meaning of an inverse trigonometric function has already been explained (see Par I, p. 297), so that a reminder only is

needed here. Thus $\sin^{-1} x$ is an inverse trigonometric function, and it is such a function that if $y = \sin^{-1} x$, then $\sin y = x$.

To differentiate $\sin^{-1} x$ with regard to x.

Let $y = \sin^{-1} x$ so that, from definition, $\sin y = x$

then $$\frac{d \sin y}{dx} = \frac{dx}{dx} = 1$$

but $$\frac{d \sin y}{dx} = \frac{d \sin y}{dy} \times \frac{dy}{dx}$$

and hence $$1 = \cos y \times \frac{dy}{dx}$$

or $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

milarly— $$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

(y being supposed to vary between o and $\frac{\pi}{2}$).

Example 14.—Find the value of $\frac{d}{dx} \tan^{-1} \frac{x}{a}$.

Let $$y = \tan^{-1} \frac{x}{a}, \textit{ i. e.,} \tan y = \frac{x}{a}$$

and $$\sec^2 y = 1 + \tan^2 y = 1 + \frac{x^2}{a^2}$$
$$= \frac{a^2+x^2}{a^2}$$

Now $$\frac{d \tan y}{dx} = \frac{d}{dx}\left(\frac{x}{a}\right) = \frac{1}{a}$$

but $$\frac{d \tan y}{dx} = \frac{d \tan y}{dy} \times \frac{dy}{dx}$$

Hence $$\frac{1}{a} = \sec^2 y \times \frac{dy}{dx}$$

or $$\frac{dy}{dx} = \frac{1}{a \sec^2 y} = \frac{1}{a} \times \frac{a^2}{a^2+x^2}$$
$$= \underline{\frac{a}{a^2+x^2}}$$

Example 15.—Find the value of $\frac{d}{dx}\cosh^{-1}\frac{x}{a}$.

Let $$y = \cosh^{-1}\frac{x}{a}$$

then $$\cosh y = \frac{x}{a}.$$

So that $$\frac{d\cosh y}{dx} = \frac{d}{dx}\left(\frac{x}{a}\right) = \frac{1}{a}$$

but $$\frac{d\cosh y}{dx} = \frac{d\cosh y}{dy}\times\frac{dy}{dx}$$

hence $$\frac{1}{a} = \sinh y\times\frac{dy}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now $\cosh^2 y - \sinh^2 y = 1$

whence $$\sinh^2 y = \cosh^2 y - 1 = \frac{x^2}{a^2} - 1$$
$$= \frac{x^2 - a^2}{a^2}$$

and $$\sinh y = \pm\frac{1}{a}\sqrt{x^2 - a^2}$$

Then, substituting this value for $\sinh y$ in (1)—

$$\frac{1}{a} = \pm\frac{1}{a}\sqrt{x^2 - a^2}\times\frac{dy}{dx}$$

or $$\frac{dy}{dx} = \pm\frac{1}{\sqrt{x^2 - a^2}}$$

i. e., $$\frac{d}{dx}\cosh^{-1}\frac{x}{a} = \pm\frac{1}{\sqrt{x^2 - a^2}}$$

Exercises 8.—On the Differentiation of a Quotient and the Differentiation of Inverse Functions.

Differentiate with respect to x the functions in Nos. 1 to 12.

1. $\frac{5x^3}{e^{7x-8}}$.
2. $\frac{\log(2-7x)}{\cos(2-7x)}$.
3. $5\sin^{-1}\frac{4x}{7}$.
4. $\cos^{-1}\frac{bx}{d^2}$.
5. $\frac{5^{2.1x}}{e^{9x-1}}$.
6. $\frac{\cosh 1{\cdot}8x}{4^{1.8x}}$.
7. $\frac{1}{\sqrt{6}}\tan^{-1}\frac{x+3}{\sqrt{6}}$.
8. $\frac{7\cos^{-1}3x}{\sqrt{1-9x^2}}$.
9. $\frac{wb(a-x)x}{2(b-x\cot B)}$.
10. $\frac{x}{a^2(a^2+x^2)^{\frac{1}{2}}}$.
11. $\frac{l^3-6l^2x+12lx^2-7x^3}{3l-4x}$ (an expression occurring in the solution of a beam problem).

12. $\dfrac{e^{\sin(1\cdot 2x+1\cdot 7)}}{\log(8x^2-7x+3)}$.

13. Assuming the results for $\dfrac{d}{dx}\cosh x$ and $\dfrac{d}{dx}\sinh x$, find the value of $\dfrac{d}{dx}\tanh x$.

Nos. 14 and 15 refer to the flow of water through circular pipes; v being the velocity of flow, Q the quantity flowing, and θ being the angle at the centre subtended by the wetted perimeter.

14. If $v = 13\cdot 1\left(1-\dfrac{\sin\theta}{\theta}\right)^{\frac{1}{2}}$ find $\dfrac{dv}{d\theta}$.

15. Given that $Q = 132\cdot 4\,\dfrac{(\theta-\sin\theta)^{\frac{5}{2}}}{\theta^{\frac{3}{2}}}$ find $\dfrac{dQ}{d\theta}$.

16. Differentiate, with respect to y, the expression—

$$\log\left(\frac{1+y}{1-y}\right)^{\frac{1}{4}}-\frac{1}{2}\tan^{-1}y.$$

17. If v (a velocity) $= r\omega\left(\sin\theta+\dfrac{\sin 2\theta}{2\sqrt{m^2-\sin^2\theta}}\right)$ and $\dfrac{d\theta}{dt}=\omega$, find the acceleration $\left(\dfrac{dv}{dt}\right)$; find also the acceleration when θ is very small.

18. If $\sin\phi = \dfrac{\sin\theta}{m}$, and $\dfrac{d\theta}{dt}=\omega$, find the angular velocity $\left(\dfrac{d\phi}{dt}\right)$ of a connecting-rod and also the angular acceleration $\left(\dfrac{d^2\phi}{dt^2}\right)$.

19. Given that $x = \dfrac{w\tan^2\theta+q}{(p-q)\tan\theta}$ find $\dfrac{dx}{d\theta}$ and hence the value of $\tan\theta$ that makes $\dfrac{dx}{d\theta}=0$.

20. Find the value of $\dfrac{dM}{dx}$ when $M = \dfrac{wx(l-x)(l-2x)}{2(3l-2x)}$. M is a bending moment, l is the length of a beam and x is a portion of that length.

21. Differentiate, with respect to t, the quotient $\dfrac{5\log(5t-8)}{t^2}$.

Partial Differentiation.—When dealing with the equation $PV = C\tau$ in connection with the theory of heat engines, we know that C alone is a constant, P, V and τ being variables. If one of these variables has a definite value, the individual values of the others are not thereby determined; *e.g.*, assuming that C and τ are known, then so also is the product PV, but not the individual values of P and V. If, now, the value of one of these is fixed, say of P, then the value of V can be calculated: therefore V depends on both P and τ, and any change in V may be due to a change in either or both of the other variables. To find the change in the value of V consequent on changes in values of P and τ,

the change in V due to the change in P (assuming that τ is kept constant) is added to the change in V due to the change in τ (P being kept constant). Rates of change found according to this plan are spoken of as *partial* rates of change, or more usually *partial derivatives*, and the process of determining them is known as *partial differentiation*.

When only two variables occur, a plane curve may be plotted to depict the connection between them, but for three variables a surface is needed. The three co-ordinate axes will be mutually at

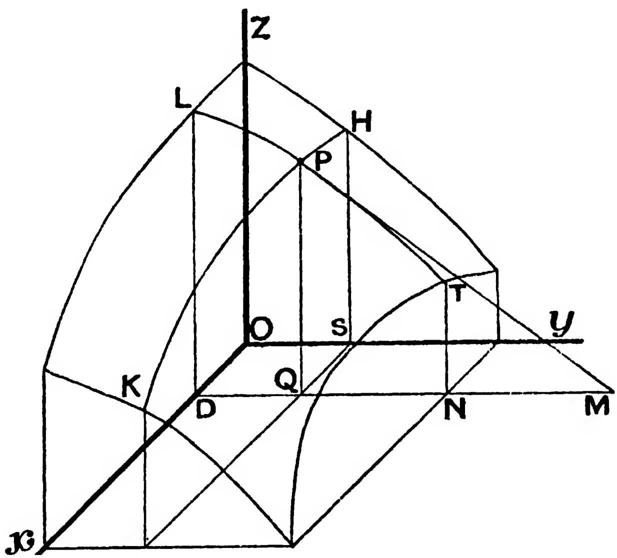


FIG. 20.

right angles, two in the plane of the paper, and the other at right angles to it. If x, y and z are the variables, we can say that z is a function of x and y, or, in the abbreviated form $z = f(x, y)$.

Similarly— $x = f(y, z)$

and $y = f(x, z)$.

Dealing with the first of these forms, and assuming the axes of x and y to be horizontal (Fig. 20), let us examine, from the aspect of the graph, the significance of this form. Giving any value to x, we know the distance of the point in front of or behind the paper: the value of y determines the distance to the right or left of the axis of z, *i. e.*, the vertical on which the point lies is

determined and the actual height up this vertical is fixed by the value of z. If z is kept constant whilst values of x and y are chosen, a number of points are found all lying on a horizontal plane, and if all such points are joined we have what is known as a contour line. Therefore, if one of the quantities is constant our work is confined to one plane; but we have already seen that when dealing with a plane, the rate of change of one quantity with regard to another is measured by the slope of a curve, hence we can ascribe a meaning to a partial derivative.

To illustrate by reference to a diagram (Fig. 20).

The point P on the surface is fixed by its co-ordinates x, y and z, or SQ, OS and QP.

If x is kept constant, the point must lie on the plane LTND. The slope of the curve LPT, as given by the tangent of the angle PMN, must measure the rate of change of z with regard to y when x is constant; and this is what we have termed the partial derivative of z with regard to y. This partial derivative may be expressed by $\frac{\partial z}{\partial y}$, or, more explicitly, by $\left(\frac{dz}{dy}\right)_x$, and if there is no possibility of ambiguity as to the quantity kept constant the suffix x may be dispensed with.

$$\left(\frac{dz}{dy}\right) = -\tan \angle \text{PMN} \quad \text{(the slope being negative, since } z \text{ decreases as } y \text{ increases).}$$

Similarly, the slope of the curve KPH

$$= \left(\frac{dz}{dx}\right) \text{ or } \frac{\partial z}{\partial x}.$$

If the variables are connected by an equation, the partial derivatives can be obtained by the use of the ordinary rules of differentiation.

Example 16.—Given that $z = 5x^2y - 2x^3y^2 + 20e^{xy}$.

Find— $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$

To find $\frac{\partial z}{\partial x}$, *i.e.*, to find the rate of change of z with regard to x when y is constant, differentiate in the ordinary way, but treating y as a constant.

Thus—
$$\frac{\partial z}{\partial x} = (5y \times 2x) - (2y^2 \times 3x^2) + 20ye^{xy}$$
$$= \underline{10xy - 6x^2y^2 + 20ye^{xy}}$$

and
$$\frac{\partial^2 z}{\partial x^2} = (10y \times 1) - (6y^2 \times 2x) + (20y \times ye^{xy})$$
$$= \underline{10y - 12xy^2 + 20y^2e^{xy}}.$$

To find $\frac{\partial z}{\partial y}$ and $\frac{\partial^2 z}{\partial y^2}$ x must be kept constant.

$$z = 5x^2y - 2x^3y^2 + 20e^{xy}$$

then
$$\frac{\partial z}{\partial y} = (5x^2 \times 1) - (2x^3 \times 2y) + 20x \, . \, e^{xy}$$
$$= \underline{5x^2 - 4x^3y + 20xe^{xy}}$$

and
$$\frac{\partial^2 z}{\partial y^2} = 0 - (4x^3 \times 1) + (20x \times x \, . \, e^{xy})$$
$$= \underline{20x^2e^{xy} - 4x^3}.$$

Example 17.—If $z = 6 \log xy - 18x^5y^2$, find the values of $\frac{\partial^2 z}{\partial x \; dy}$ and $\frac{\partial^2 z}{\partial y \; \partial x}$, and state the conclusion to be drawn from the results.

To find $\frac{\partial^2 z}{\partial x \; \partial y}$ we must first find the value of $\frac{\partial z}{\partial y}$, x being regarded as a constant: then if Y be written for this expression the value of $\frac{\partial Y}{\partial x}$ must next be determined, y being treated as a constant, and this is the value of $\frac{\partial^2 z}{\partial x \; \partial y}$.

Now $\frac{\partial z}{\partial y} = \frac{6 \times x}{xy} - 18x^5 \times 2y = \frac{6}{y} - 36x^5y = \text{Y, say.}$

Differentiating this expression with regard to x, y being regarded as a constant—

$$\frac{\partial Y}{\partial x} = 0 - 180x^4y$$

or
$$\frac{\partial^2 z}{\partial x \; \partial y} = \underline{-180x^4y.}$$

Now
$$\frac{\partial z}{\partial x} = \left(\frac{6 \times y}{xy}\right) - (18y^2 \times 5x^4) = \frac{6}{x} - 90y^2x^4$$

and
$$\frac{\partial}{\partial y}\frac{\partial z}{\partial x} \text{ or } \frac{\partial^2 z}{\partial y \; \partial x} = 0 - (90x^4 \times 2y) = \underline{-180x^4y,}$$

i. e.,
$$\frac{\partial^2 z}{\partial x \; \partial y} = \frac{\partial^2 z}{\partial y \; \partial x}.$$

Hence the order of differentiation does not affect the result.

Total Differential.—If y is a function of x, then $y = f(x)$ and $\frac{dy}{dx} = \frac{d}{dx}f(x)$ or $f'(x)$. Then $\delta y = f'(x)\delta x$ (approximately)* and $dy = f'(x)dx$ (accurately).

dy and dx are spoken of as *differentials*, and $f'(x)$ is the coefficient of the differential dx; hence we see the reason for the term *differential coefficient.*

* Cf. Fig. 26, p. 107, where QM = slope of tangent at P $\times$ δx (approximately).

If z is a function of x and y, *i. e.*, $z = f(x, y)$, the total differential dz is obtained from the partial differentials dx and dy by the use of the following rule—

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

The reason for this is more clearly seen if we work from the fundamental idea of rates of change, and introduce the actually measurable quantities like δz, δx and δy.

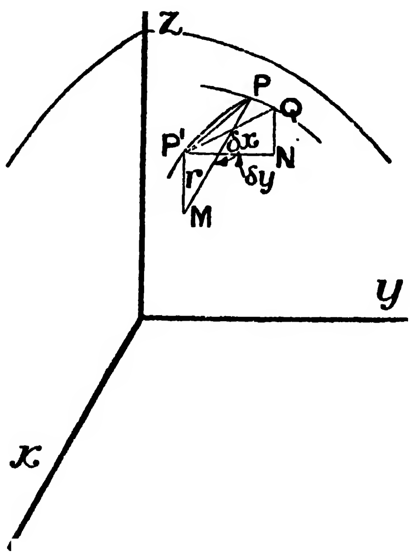


FIG. 21.

Thus— $$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y.$$

or total change in z = change in z due to the change in x + change in z due to the change in y.

The change in z due to the change in x must be measured by the product of the change in x multiplied by the rate at which z is changing with regard to x; and this fact can be better illustrated by reference to a diagram (Fig. 21).

Let P be a point (x, y, z) on a surface, and let P move to a new position Q near to P. The change of position is made up of—

(*a*) A movement δx to P′ on the surface (y being kept constant).

(*b*) A movement δy to Q on the surface (x being kept constant).

In (*a*) z increases by MP′

and $MP' = PM \times \tan \angle MPP'$

$$= \delta x \times \text{mean value of } \frac{\partial z}{\partial x}.$$

In (*b*) the change in $z = NQ$

$$= \delta y \times \text{mean value of } \frac{\partial z}{\partial y}.$$

If P, P′ and Q are taken extremely close to one another, the mean or average slopes become the actual slopes and

the total change in $z = \delta z$

$$= MP' + NQ = \delta x \frac{\partial z}{\partial x} + \delta y \frac{\partial z}{\partial y}.$$

Example 18.—If Kinetic Energy $= K = \frac{wv^2}{2g}$, find the change in the energy as w changes from 49 to 49·5 and v from 1600 to 1590.

From the above rule, the change in $K = \delta K$

$$= \delta w \frac{\partial K}{\partial w} + \delta v \frac{\partial K}{\partial v}.$$

Now $\delta w = 49{\cdot}5 - 49 = {\cdot}5$

and $\delta v = 1590 - 1600 = -10.$

Also $\frac{\partial K}{\partial w}$ (*i. e.*, v being constant) $= \frac{d}{dw}\left(\frac{v^2}{2g} \times w\right) = \frac{v^2}{2g} \times 1$

and $\frac{\partial K}{\partial v}$ (w being constant) $= \frac{d}{dv}\left(\frac{w}{2g} \times v^2\right) = \frac{w}{2g} \times 2v.$

$$\therefore \quad \delta K = {\cdot}5 \times \frac{v^2}{2g} + (-10) \times 2\frac{vw}{2g}.$$

$$= \frac{{\cdot}5 \times 1600^2}{64{\cdot}4} - \frac{20 \times 1600 \times 49}{64{\cdot}4}$$

$$= 19880 - 24380 = \underline{-4500 \text{ units.}}$$

Example 19.—A quantity of water Q is measured by

$$Q = C\frac{D^2}{4}\sqrt{2gH}.$$

If $r_1 =$ the probable error of D, a diameter, $r_2 =$ the probable error of H, a head, and R = the probable error of Q,

$$R = \sqrt{r_1^2\left(\frac{dQ}{dD}\right)^2 + r_2^2\left(\frac{dQ}{dH}\right)^2}$$

where $\left(\frac{dQ}{dD}\right)$ and $\left(\frac{dQ}{dH}\right)$ are partial derivatives.

Find an expression for R.

$$\left(\frac{dQ}{dD}\right)_H = \frac{d}{dD}\left(C \times \frac{\pi}{4}\sqrt{2gH} \, . \, D^2\right)$$

$$= 2D \times C \times \frac{\pi}{4}\sqrt{2gH}.$$

Also—
$$\left(\frac{dQ}{dH}\right)_D = \frac{d}{dH}\left(C \times \frac{\pi}{4} D^2 \sqrt{2g} \cdot H^{\frac{1}{2}}\right)$$
$$= \frac{1}{2H^{\frac{1}{2}}} \times \frac{\pi CD^2}{4} \sqrt{2g}$$

Hence—
$$R = \sqrt{\frac{r^2{}_1 C^2 D^2 \pi^2 \times 2gH}{4} + \frac{r_2{}^2 \times \pi^2 C^2 D^4 \times 2g}{64H}}$$
$$= C \cdot \frac{\pi}{4} D^2 \cdot \sqrt{2gH} \cdot \sqrt{r_1{}^2 \frac{4}{D^2} + r_2{}^2 \frac{1}{4H^2}}$$
$$= Q\sqrt{4\left(\frac{r_1}{D}\right)^2 + \frac{1}{4}\left(\frac{r_2}{H}\right)^2}.$$

or
$$\frac{R}{Q} = \sqrt{4\left(\frac{r_1}{D}\right)^2 + \frac{1}{4}\left(\frac{r_2}{H}\right)^2}$$

i. e., if the probable error of D is 3% and that of H is 1% that of $Q = \sqrt{4\,(\cdot 03)^2 + \frac{1}{4}(\cdot 01)^2}$
$= \cdot 0602$, *i. e.*, is about 6%.

Logarithmic Differentiation.—Occasionally it is necessary to differentiate an expression which can be resolved into a number of factors; and in such a case, to avoid repeated applications of the rules for the differentiation of products and quotients, we may first take logs throughout, and then differentiate, making use of the rule for the differentiation of a function of a function. By the judicious use of this artifice much labour can often be saved.

Example 20.—Find the value of $\frac{d}{dx}\left\{\frac{(3x-4)(4x+7)}{(2x-9)}\right\}$.

Let—
$$y = \frac{(3x-4)(4x+7)}{(2x-9)}$$

then $\log y = \log(3x-4) + \log(4x+7) - \log(2x-9)$.

Differentiating with regard to x—
$$\frac{d \log y}{dx} = \frac{3}{(3x-4)} + \frac{4}{(4x+7)} - \frac{2}{(2x-9)}$$

but
$$\frac{d \log y}{dx} = \frac{d \log y}{dy} \times \frac{dy}{dx} = \frac{1}{y} \cdot \frac{dy}{dx}$$

so that
$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{3}{(3x-4)} + \frac{4}{(4x+7)} - \frac{2}{(2x-9)}$$

or
$$\frac{dy}{dx} = \frac{(3x-4)(4x+7)}{(2x-9)} \times$$
$$\left\{\frac{24x^2 - 66x - 189 + 24x^2 + 144 - 140x - 24x^2 - 10x + 56}{(3x-4)(4x+7)(2x-9)}\right\}$$
$$= \frac{24x^2 - 216x + 11}{(2x-9)^2}.$$

[As an exercise, the reader should work this according to the following plan. Write $y = \frac{12x^2+5x-28}{(2x-9)}$, and then use the rule for the differentiation of a quotient.]

It is with examples in which powers of factors occur that this method is most useful.

Example 21.—Find $\frac{dy}{dx}$ when $y = \frac{(7x+2)^3(x-1)}{(2x-5)^2}$.

Taking logs throughout—

$$\log y = 3 \log (7x+2) + \log (x-1) - 2 \log (2x-5)$$

Then—

$$\frac{d \log y}{dx} = \frac{3\times 7}{(7x+2)} + \frac{1}{(x-1)} - \frac{2\times 2}{(2x-5)}$$

$$= \frac{42x^2+105-147x+14x^2-31x-10-28x^2+20x+8}{(7x+2)(x-1)(2x-5)}$$

$$= \frac{28x^2-158x+103}{(7x+2)(x-1)(2x-5)}$$

$$\therefore \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{28x^2-158x+103}{(7x+2)(x-1)(2x-5)}$$

$$i.e., \quad \frac{dy}{dx} = \frac{(7x+2)^3(x-1)}{(2x-5)^2} \times \frac{28x^2-158x+103}{(7x+2)(x-1)(2x-5)}$$

$$= \frac{(7x+2)^2(28x^2-158x+103)}{(2x-5)^3}.$$

Exercises 9.—On Partial Differentiation and Logarithmic Differentiation.

1. In measuring the sides of a rectangle, the probable errors in the sides were r_1 and r_2. If A = area and a and b are the sides, find the probable error R in A.

Given that— $$R = \sqrt{r_1^2\left(\frac{dA}{da}\right)^2 + r_2^2\left(\frac{dA}{db}\right)^2}$$

the derivatives being partial.

2. If $z = 3{\cdot}26x^5e^{2y}$, find $\frac{\partial z}{\partial y}$ and $\frac{\partial^2 z}{\partial x^2}$.

3. If $s = t^{5.2} - pt^2 + \log (5p-3) \times e^{4t}$, find $\frac{\partial s}{\partial p}$, and $\frac{\partial s}{\partial t}$.

4. If $v = (4-u)^3(3+8u)^2$, find $\frac{dv}{du}$.

5. If $y = \frac{(1+x)^n}{(1-x)^n}$, show that $\frac{dy}{dx} = \frac{2ny}{1-x^2}$.

6. If $y = 8x(1 \cdot 7 + \cdot 2x)^4$, find $\dfrac{dy}{dx}$.

7. Differentiate, with respect to x, $\dfrac{x+3}{(x^2+6x+5)^2}$

8. Find the rate of discharge $\dfrac{dm}{dt}$ of air from a closed reservoir when $m = \dfrac{pv}{c\tau}$, m, p, v and τ all being variables.

9. If $x = r\cos\theta$, $y = r\sin\theta$, and u is a function of both x and y, prove that—

$$\left(\frac{du}{dx}\right)_y = \cos\theta\left(\frac{du}{dr}\right)_\theta - \frac{1}{r}\sin\theta\left(\frac{du}{d\theta}\right)_r$$

and
$$\left(\frac{du}{dy}\right)_x = \sin\theta\left(\frac{du}{dr}\right)_\theta + \frac{1}{r}\cos\theta\left(\frac{du}{d\theta}\right)_r.$$

10. If $V = 4e^{2x-3t} - 5e^{2x+3t}$ show that $4\dfrac{\partial^2 V}{\partial t^2} = 9\dfrac{\partial^2 V}{\partial x^2}$.

CHAPTER IV

APPLICATIONS OF DIFFERENTIATION

HAVING developed the rules for the differentiation of the various functions, algebraic and trigonometric, we are now in a position to apply these rules to the solution of practical problems. By far the most important and interesting direction in which differentiation proves of great service is in the solution of problems concerned with maximum and minimum values; and with these problems we shall now deal.

Maximum and Minimum Values.—Numerous cases present themselves, both in engineering theory and practice, in which the value of one quantity is to be found such that another quantity, which depends on the first, has a maximum or minimum value when the first has the determined value.

E. g., suppose it is desired to arrange a number of electric cells in such a way that the greatest possible current is obtained from them. Knowing the voltage and internal resistance of each cell and the external resistance through which the current is to be passed, it is possible by simple differentiation to determine the relation that must exist between the external resistance and the total internal resistance in order that the maximum current flows.

Again, it might be necessary to find the least cost of a hydraulic installation to transmit a certain horse-power. Here a number of quantities are concerned, such as diameter of piping, price of power, length of pipe line, etc., any one of which might be treated as the main variable. By expressing all the conditions in terms of this one variable and proceeding according to the plan now to be demonstrated, the problem would become one easy of solution.

A graphic method for the solution of such problems has already been treated very fully (see Part I, pp. 183 *et seq.*). This method, though direct and perfectly general in its application, is somewhat laborious, and unless the graphs are drawn to a large scale in the neighbourhood of the turning points, the results obtained are usually good approximations only. In consequence of these failings

of the graphic treatment, the algebraic method is introduced, but it should be remembered that its application is not so universal as that of the solution by plotting.

The theory of the algebraic method can be simply explained in the following manner :—

The slope of a curve measures the rate of change of the ordinate with regard to the abscissa; and hence, when the slope of the

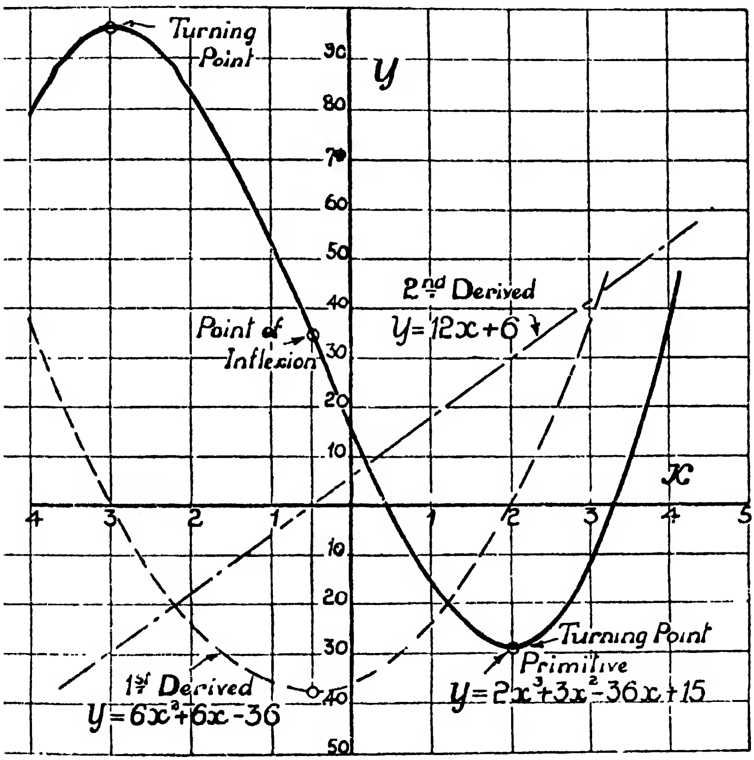


FIG. 22.—Maximum and Minimum Values.

curve is zero, the rate of change of the function is zero, and the function must have a turning value, which must be either a maximum or a minimum. But it has already been pointed out that the slope of a curve is otherwise defined as the derivative or the differential coefficient of the function; therefore the function has a turning value whenever its derivative is zero, provided that the derivative changes sign in the immediate neighbourhood.

Hence, to find maximum or minimum values of a function we must first determine the derivative of the function, and then find

the value or values of the I.V. which make the derivative zero; the actual maximum or minimum values of the function being found by the substitution of the particular values of the I.V. in the expression for the function.

The rule, stated in a concise form, is: **To find the value of the I.V. which makes the function a maximum or minimum, differentiate the function, equate to zero and solve the resulting equation.**

The full merit of the method will be best appreciated by the discussion of a somewhat academic problem before proceeding to some of a more practical nature.

Example 1.—Find the values of x which give to the function $y = 2x^3+3x^2-36x+15$ maximum or minimum values. Find also the value of x at the point of inflexion of the curve.

This question may be treated from two points of view, viz.—

(*a*) *From the graphical aspect.*

We first plot the primitive curve $y = 2x^3+3x^2-36x+15$ (see Fig. 22), the table of values for which is :—

x	x^2	x^3	$2x^3+3x^2-36x+15$	y
−4	16	−64	−128 +48+144+15	79
−3	9	−27	− 54+27+108+15	96
−2	4	− 8	− 16+12+ 72+15	83
− 1	1	− 1	− 2+ 3+ 36+ 15	52
0	0	0	0+ 0− 0+15	15
1	1	1	2+ 3− 30+15	−16
2	4	8	16+12− 72+15	−29
3	9	27	54+27−108+15	−12
4	16	64	128+48−144+15	47
5	25	125	250+75−180+15	160

This curve has two turns and two turns only, and consequently y has two turning values, one being a maximum and one a minimum. By successive graphic differentiation the first and second derived curves may be drawn, these being shown on the diagram.

Now for values of x less than -3 the slope of the primitive curve is positive, as is demonstrated by the fact that the ordinates of the first derived curve are positive. At $x = -3$ the primitive curve is horizontal and the first derived curve crosses the x-axis; and since the ordinates of the first derived curve give the values of $\frac{dy}{dx}$, we see that when the primitive curve has a turning value, the value of $\frac{dy}{dx} = 0$. For values of x between -3 and $+2$ the slope of the primitive is negative; when $x = +2$ the slope is zero, and from that

point the slope is positive. Thus y has turning values when $x = -3$ and when $x = +2$; these values being a maximum at $x = -3$ and a minimum at $x = +2$ as observed from the curve.

This investigation proves of service when we proceed to treat the question from the algebraic aspect; in fact, for complete understanding the two methods must be interwoven.

(*b*) *From the algebraic point of view.*

Let $$y = 2x^3+3x^2-36x+15$$

then $$\frac{dy}{dx} = 6x^2+6x-36$$
$$= 6(x^2+x-6).$$

Now in order that y may have turning values we have seen that it is necessary that $\frac{dy}{dx} = 0$.

But $$\frac{dy}{dx} = 0 \text{ if } 6(x^2+x-6) = 0$$

i. e., $$\text{if } 6(x+3)(x-2) = 0$$

i. e., $$\text{if } x = -3 \text{ or } 2$$

and hence y has turning values when $x = -3$ and $x = +2$. We do not yet, however, know the character of these turning values, so that our object must now be to devise a simple method enabling us to discriminate between values of x giving maximum and minimum values to y.

An obvious, but slow, method is as follows: Let us take a value of x slightly less than -3, say $-3{\cdot}1$; then the calculated value of y is $95{\cdot}85$. Next, taking a value of x rather bigger than -3, say $-2{\cdot}9$, the value of y is found to be $95{\cdot}85$. Therefore, as x increases from $-3{\cdot}1$ to -3 and thence to $-2{\cdot}9$, y has the values $95{\cdot}85$, 96, and $95{\cdot}85$ respectively. Thus the value of y must be a maximum at $x = -3$, since its values on either side are both less than its value when $x = -3$. In like manner it can be shown that when $x = +2$, y has a minimum value.

The arithmetical work necessary in this method can, however, be dispensed with by the use of a more mathematical process, now to be described.

Referring to the first derived curve, the equation of which is $y = 6x^2+6x-36$, we note that as x increases from -4 to -3 the ordinate of the derived curve decreases from 36 to 0; from $x = -3$ to $x = -{\cdot}5$ the ordinate is negative but increasing numerically, *i. e.*, in the neighbourhood of $x = -3$ the slope of the second derived curve, which is the slope curve of the first derived curve, is negative (for the ordinate decreases as the abscissa or the I.V. increases). But the slope of the first derived

curve, and thus the ordinate of the second derived curve, must be expressed by $\frac{d^2y}{dx^2}$, so that we conclude that in the neighbourhood of a maximum value of the original function the second derivative of it is a negative quantity.

In the same way we see that in the neighbourhood of a minimum value of the function, its second derivative is a positive quantity. Hence a more direct method of discrimination between the turning values presents itself: **Having found the values of the I.V. causing turning values of the original function, substitute these values in turn in the expression for the second derivative of the function; if the result is a negative, then the particular value of the I.V. considered is that giving a maximum value of the function and vice-versa.**

This rule may be expressed in the following brief fashion:—

Let $y = f(x)$ and let the values of x that make $\frac{df}{dx}(x)$ or $f'(x) = 0$ be x_1 and x_2.

Find the value of $\frac{d^2y}{dx^2}$ or $f''(x)$, as it may be written, and in this expression substitute in turn the values x_1 and x_2 in place of x: the values thus obtained are those of $f''(x_1)$ and $f''(x_2)$ respectively. Then if $f''(x_1)$, say, is negative, y has a maximum value when $x = x_1$; and if $f''(x_1)$ is positive, y has a minimum value when $x = x_1$.

Applying to our present example:—

$$y = 2x^3+3x^2-36x+15$$

$$f'(x) = \frac{dy}{dx} = 6x^2+6x-36$$

$$f''(x) = \frac{d^2y}{dx^2} = 12x+6.$$

When $x = -3$ the value of $\frac{d^2y}{dx^2}$ is $12(-3)+6$, *i. e.*, $f''(-3) = -30$; and since $f''(-3)$ is a negative quantity, y is a maximum when $x = -3$.

Similarly, $f''(+2) = 12(2)+6 = +30$

and hence y is a minimum when $x = +2$.

Referring to the second derived curve, *i. e.*, the curve $y = 12x+6$, we note that its ordinate is negative for all values of x less than $-\cdot5$ and positive for all values of x greater than $-\cdot5$, the curve crossing the axis of x when $x = -\cdot5$. This indicates that when $x = -\cdot5$ the first derived curve has a turning value; but the first

derived curve is the curve of the gradients of the primitive curve, and hence when $x = -\cdot 5$ the *gradient* of the primitive must have a turning value, which may be either a maximum or a minimum. In other words, if we had placed a straight edge to be tangential in all positions to the primitive curve, it would rotate in a right-handed direction until $x = -\cdot 5$ was reached, after which the rotation would be in the reverse direction. A point on the curve at which the gradient ceases to rotate in the one direction and commences to rotate in the opposite direction is called a ***point of inflexion*** of the curve. Thus points of inflexion or contra-flexure occur when $\frac{d^2y}{dx^2} = 0$, provided that $\frac{d^2y}{dx^2}$ changes sign.

A useful illustration of the necessity for determining points of contra-flexure is furnished by cases of fixed beams. We have

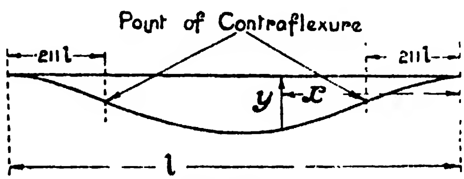


FIG. 23.

already seen that the bending moment at any section is proportional to the value of $\frac{d^2y}{dx^2}$ there; hence there must be points of contra-flexure when the bending moment is zero.

Example 2.—Find the positions of the points of contra-flexure of a beam fixed at its ends and uniformly loaded with w units per foot; the deflected form having the equation—

$$y = \frac{1}{\mathrm{EI}}\left(\frac{wlx^3}{12} - \frac{wl^2x^2}{24} - \frac{wx^4}{24}\right).$$

We may regard this question from either the graphic aspect or the physical. According to the former we see that it is necessary to determine the points of inflexion, and therefore to find values of x for which $\frac{d^2y}{dx^2}$ is zero.

Reasoning from the physical basis we arrive at the same result, by way of the following argument: the bending moment, which is expressed by $\mathrm{EI}\frac{d^2y}{dx^2}$, changes sign, as is indicated by the change in the curvature of the beam (see Fig. 23), and therefore at two points the bending moment must be zero, since the variation in it is uniform

and continuous; but the bending moment is zero when $\frac{d^2y}{dx^2}$ is zero, since $M = EI\frac{d^2y}{dx^2}$.

Now
$$y = \frac{w}{EI}\left(\frac{lx^3}{12} - \frac{l^2x^2}{24} - \frac{x^4}{24}\right),$$

hence
$$\frac{dy}{dx} = \frac{w}{EI}\left[\left(\frac{l}{12}\times 3x^2\right) - \left(\frac{l^2}{24}\times 2x\right) - \frac{4x^3}{24}\right]$$
$$= \frac{w}{EI}\left(\frac{lx^2}{4} - \frac{l^2x}{12} - \frac{x^3}{6}\right)$$

and
$$\frac{d^2y}{dx^2} = \frac{w}{EI}\left[\left(\frac{l}{4}\times 2x\right) - \left(\frac{l^2}{12}\times 1\right) - \frac{3x^2}{6}\right]$$
$$= \frac{w}{EI}\left(\frac{lx}{2} - \frac{l^2}{12} - \frac{x^2}{2}\right).$$

Now the bending moment $M = EI \cdot \frac{d^2y}{dx^2} = w\left(\frac{lx}{2} - \frac{l^2}{12} - \frac{x^2}{2}\right)$

and $M = 0$ if $\frac{lx}{2} - \frac{l^2}{12} - \frac{x^2}{2} = 0$, *i. e.* if $6lx - l^2 - 6x^2 = 0$,

i. e., if $6x^2 - 6lx + l^2 = 0$

or
$$x = \frac{6l \pm \sqrt{36l^2 - 24l^2}}{12}$$
$$= \frac{l}{12}(6 \pm 3 \cdot 46)$$
$$= \underline{\cdot 789l \text{ or } \cdot 211l.}$$

Hence the points of inflexion occur at points distant ·211 of the length from the ends.

Example 3.—A line, 5 ins. long, is to be divided into two parts such that the square of the length of one part together with four times the cube of the length of the other is a minimum. Find the position of the point of section.

Let x ins. = the length of one part, then $5-x$ = length of the other part.

Then $(5-x)^2+4x^3$ is to be a minimum.

Let
$$y = (5-x)^2+4x^3$$
$$= 25+x^2-10x+4x^3.$$

Then
$$\frac{dy}{dx} = 2x-10+12x^2$$
$$= 2(6x^2+x-5)$$
$$= 2(6x-5)(x+1).$$

Hence $\frac{dy}{dx} = 0$ if $x = \frac{5}{6}$ or -1 (the latter root implying external cutting).

To test for the nature of the turning value—

$$\frac{dy}{dx} = 12x^2+2x-10$$

and
$$\frac{d^2y}{dx^2} = 24x+2.$$

When
$$x = \frac{5}{6}$$

$$\frac{d^2y}{dx^2} = \left(\frac{24\times 5}{6}\right)+2 = \text{a positive quantity.}$$

Therefore y is a minimum when $x = \frac{5}{6}$ and the required point of section is $\frac{5}{6}$ in. from one end.

Example 4.—If s = equivalent parasite area of an airplane
S = wing area
K = lifting efficiency

T, the thrust required, is $T = C\left(i+\frac{1}{f^2 i}\right)$ where C is a constant, i the wing incidence expressed in radians, and f is obtained from the formula $f^2 = KS/0{\cdot}08s$.

Taking $S = 25s$ and $K = 0{\cdot}4$, find the angle of incidence for the case when the least thrust is required.

$$f^2 = \frac{KS}{\cdot 08s} = \frac{\cdot 4\times 25}{\cdot 08} = 125.$$

The thrust $T = C\left(i+\frac{1}{f^2 i}\right)$ and since i is the only variable in this expression, we must differentiate with regard to it.

Thus
$$\frac{dT}{di} = C\left(1-\frac{1}{f^2 i^2}\right)$$

and
$$\frac{dT}{di} = 0 \text{ if } \left(1-\frac{1}{f^2 i^2}\right) = 0$$

i. e.,
$$\text{if } i^2 = \frac{1}{f^2} = \frac{1}{125}.$$

Thus
$$i = \cdot 0895$$

or the thrust required is either a maximum or minimum when the angle of incidence is $\cdot 0895$ radian.

To test whether this turning value is a maximum or a minimum, let us find the second derivative—

$$\frac{dT}{di} = C\left(1-\frac{1}{f^2 i^2}\right)$$

$$\frac{d^2T}{di^2} = C\left(0+\frac{2}{f^2 i^3}\right).$$

When $i = \cdot 0895$, $\frac{d^2T}{di^2}$ must be positive, and hence T has its minimum value when $i = \cdot 0895$.

Example 5.—Find the dimensions of the greatest cylinder that can be inscribed in a right circular cone of height 6 ins. and base 10 ins. diameter.

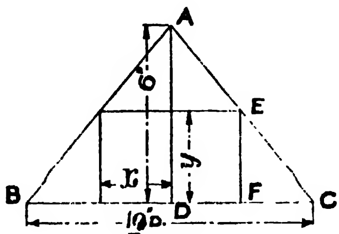


FIG. 24.

Assume that the radius of the base of the cylinder $= x$ ins. (Fig. 24)
and the height of the cylinder $= y$ ins.

Then the volume $= V = \pi x^2 y$.

We must, then, obtain an expression for y in terms of x before differentiating with regard to x.

From the figure, by similar triangles, taking the triangles ADC and EFC—

$$\frac{6}{5} = \frac{y}{5-x}$$

or $$y = \frac{6}{5}(5-x)$$

Hence $$V = \pi x^2 \times \frac{6}{5}(5-x) = \frac{6\pi}{5}(5x^2-x^3)$$

and $$\frac{dV}{dx} = \frac{6\pi}{5}(10x-3x^2)$$

Thus $$\frac{dV}{dx} = 0 \text{ if } x(10-3x) = 0$$

i. e., if $x = 0$ (giving the cylinder of zero volume)
or if $10 = 3x$, *i. e.*, $x = \underline{3\frac{1}{3} \text{ ins.}}$

Then $$y = \frac{6}{5}(5-3\tfrac{1}{3}) = \underline{2 \text{ ins.}}$$

and the volume of the greatest cylinder $= \pi \times \left(\frac{10}{3}\right)^2 \times 2 = \underline{69{\cdot}8 \text{ cu. ins.}}$

Example 6.—The total running cost in pounds sterling per hour of a certain ship being given by—

$$C = 4{\cdot}5 + \frac{v^3}{2100}$$

where v = speed in knots, find for what speed the total cost for a journey is a minimum.

The total cost for the journey depends on—

(*a*) The cost per hour; and
(*b*) The number of hours taken over the journey.

Item (*b*) depends inversely on the speed, so that if the journey were 2000 nautical miles the time taken would be $\frac{2000}{v}$ hours; or, in general, the number of hours $= \frac{K}{v}$.

Then the total cost for a journey of K nautical miles

$$= C_t = \frac{K}{v}\times\left(4{\cdot}5+\frac{v^3}{2100}\right)$$

$$= K\left(4{\cdot}5v^{-1}+\frac{v^2}{2100}\right).$$

Differentiating with regard to the variable v

$$\frac{dC_t}{dv} = K\left[(4{\cdot}5\times-1\times v^{-2})+\frac{2v}{2100}\right]$$

$$= K\left(-\frac{4{\cdot}5}{v^2}+\frac{v}{1050}\right)$$

Then $\quad \frac{dC_t}{dv} = 0$ if $-\frac{4{\cdot}5}{v^2}+\frac{v}{1050} = 0$

i. e., $\quad$ if $\frac{v}{1050} = \frac{4{\cdot}5}{v^2}$

or $\quad v^3 = 4{\cdot}5\times1050 = 4725$

hence $\quad \underline{v = 16{\cdot}78 \text{ knots}}.$

Example 7.—A water main is supplied by water under a head of 60 ft. The loss of head due to pipe friction, for a given length, is proportional to the velocity squared. Find the head lost in friction when the horse-power transmitted by the main is a maximum.

If v = velocity of flow, then—

Head lost $= Kv^2$, where K is some constant,

i. e., the effective head $= 60-Kv^2 = H_e$.

$$\text{H.P. transmitted} = \frac{\text{Quantity (in lbs. per min.)}\times\text{effective head (in feet)}}{33000}$$

$$= \frac{\text{area (in sq. ft.)}\times\text{velocity (ft. per min.)}\times62{\cdot}4\times H_e}{33000}$$

$= CvH_e$, where C is some constant

$$= Cv(60-Kv^2)$$

$$= C(60v-Kv^3)$$

Then $\quad \frac{d}{dv}(\text{H.P.}) = C(60-3Kv^2)$

$$= C(60-180+3H_e)$$

or $\quad \frac{d}{dv}(\text{H.P.}) = 0$ when $3H_e = 120$

i. e., $\quad H_e = 40.$

In general, then, the maximum horse-power is transmitted when the head lost is one-third of the head supplied, *i. e.*, the maximum efficiency is $\frac{2}{3}$ or 66·7%.

Example 8.—The stiffness of a beam is proportional to the breadth and the cube of the depth of the section. Find the dimensions of the stiffest beam that can be cut from a cylindrical log 4 ins. in diameter.

From hypothesis $\qquad S \propto bd^3$

or $\qquad S = Kbd^3.$

Both breadth and depth will vary, but they depend on each other; and from Fig. 25 we see that $b^2 = 16 - d^2$. Hence we can substitute for b its value in terms of d and then differentiate with regard to d; accordingly—

$$S = Kd^3\sqrt{16-d^2}.$$

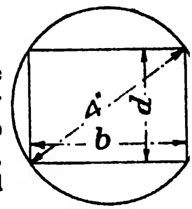


FIG. 25.

As it stands this would be a rather cumbersome expression to differentiate, and we therefore employ a method which is often of great assistance. Since we are dealing with positive quantities throughout, S^2 will be a maximum when S is a maximum,* and hence we square both sides before differentiating.

Thus— $\qquad S^2 = K^2d^6(16-d^2) = K^2(16d^6-d^8)$

and $\qquad \dfrac{dS^2}{d.d} = K^2(96d^5-8d^7) = 8d^5(12-d^2)$

Hence— $\qquad \dfrac{dS^2}{d.d} = 0$ if $d^5 = 0$, *i. e.*, $d = 0$ (giving zero stiffness)

or if— $\qquad d^2 = 12$

i. e., $\qquad d = \underline{3{\cdot}464}$ ins.

Hence $\qquad b = \sqrt{16-12} = \underline{2 \text{ ins.}}$

---

* If we were dealing with negative quantities it would be incorrect to say that the quantity itself had a maximum value when its square was a maximum, for suppose the values of the quantity y in the neighbourhood of its maximum value were -13, -12, -11, -10, -11, -12, etc., corresponding values of y^2 would be $+169$, $+144$, $+121$, $+100$, $+121$, $+144$, so that if $y = -10$ (its maximum value) when $x = 4$, say, then $y^2 = 100$ when $x = 4$, and therefore a minimum value of y^2 occurs when $x = 4$, and not a maximum.

Example 9.—Find the shape of the rectangular channel of given sectional area A which will permit the greatest flow of water; being given that $Q = Av$, $v = c\sqrt{mi}$, m = hydraulic mean depth = $\dfrac{\text{area}}{\text{wetted perimeter}}$ and i is the hydraulic gradient; Q being the quantity flowing.

Let the breadth of the section be b and the depth d; then, by hypothesis—

$$bd = A, \text{ whence } b = \frac{A}{d}.$$

$$m = \frac{\text{area}}{\text{wetted perimeter}} = \frac{A}{b+2d} \text{ and therefore } v = c\sqrt{i}\sqrt{\frac{A}{b+2d}}$$

$$= c\sqrt{Ai}\,.\,\frac{1}{\sqrt{b+2d}}$$

Hence—
$$Q = Av = Ac\sqrt{Ai}\,.\,\frac{1}{\sqrt{b+2d}}$$

$$= K\,.\,\frac{1}{\sqrt{b+2d}} \text{ where } K = Ac\sqrt{Ai}$$

Q will be a maximum when Q^2 is a maximum, hence we shall find the value of b for which Q^2 is a maximum.

$$Q^2 = K^2\frac{1}{b+2d} = K^2\,.\,\frac{1}{b+\frac{2A}{b}}.$$

Also Q^2 is a maximum when the denominator of this fraction is a minimum.

Let this denominator be denoted by D—

then
$$\frac{dD}{db} = \frac{d}{db}\left(b+\frac{2A}{b}\right) = 1-\frac{2A}{b^2}$$

and
$$\frac{dD}{db} = 0 \quad \text{if} \quad 1 = \frac{2A}{b^2}, \quad i.e., \text{ if } \quad b = \sqrt{2A}.$$

Now
$$d = \frac{A}{b} = \frac{A}{\sqrt{2A}} = \sqrt{\frac{A}{2}}.$$

∴ the dimensions would be—

$$\text{depth} = \underline{\sqrt{\frac{A}{2}}} \text{ and } \underline{\text{breadth} = \sqrt{2A}.}$$

Example 10.—For a certain steam engine the expression for W, the brake energy per cu. ft. of steam, was found in terms of r, the ratio of expansion, as follows—

$$W = \frac{120\left(\frac{1+\log r}{r}\right)-27}{\frac{\cdot 00833}{r}+\cdot 000903}$$

Find the value of r that makes W a maximum.

Before proceeding to differentiate, we can put the expression in a somewhat simpler form.

Thus—
$$W = \frac{120(1+\log r)-27r}{\cdot 00833+\cdot 000903r}$$

and W is a quotient $= \frac{u}{v}$ where $u = 120(1+\log r)-27r$

so that $$\frac{du}{dr} = \frac{120}{r} - 27$$

and $$v = \cdot 00833 + \cdot 000903r$$

so that $$\frac{dv}{dr} = \cdot 000903.$$

Hence $$\frac{dW}{dr} = \frac{v\frac{du}{dr} - u\frac{dv}{dr}}{v^2}$$

$$= \frac{(\cdot 00833 + \cdot 000903r)\left(\frac{120}{r} - 27\right) - [120(1 + \log r) - 27r]\cdot 000903}{(\cdot 00833 + \cdot 000903r)^2}$$

Now $\frac{dW}{dr} = 0$ if the numerator of the right-hand side $= 0$

i.e., if $$\left(\frac{\cdot 00833 \times 120}{r}\right) - (27 \times \cdot 00833) + (120 \times \cdot 000903) - (27 \times \cdot 000903r)$$
$$- (120 \times \cdot 000903) - (120 \times \cdot 000903 \log r) + (27 \times \cdot 000903r) = 0$$

i.e., if $$\frac{1}{r} - \cdot 225 - \cdot 1084 \log r = 0.$$

This equation must be solved by plotting, the intersection of the curves $y_1 = \cdot 1084 \log r$ and $y_2 = \frac{1}{r} - \cdot 225$ being found; the value of r here being 2·93.

Hence— $$\underline{r = 2\cdot 93.}$$

Example 11.—The value of a secondary electric current was given by the formula—

$$y = \frac{I}{2}\left(e^{-\frac{Rt}{L+M}} - e^{-\frac{Rt}{L-M}}\right)$$

where L = inductance of primary circuit
R = resistance of primary circuit
M = coefficient of mutual inductance
I = steady current.

Find for what value of t, y has a maximum value.

$$y = \frac{I}{2}\left(e^{-\frac{Rt}{L+M}} - e^{-\frac{Rt}{L-M}}\right)$$

Then $$\frac{dy}{dt} = \frac{I}{2}\left(-\frac{R}{L+M}e^{-\frac{Rt}{L+M}} + \frac{R}{L-M}e^{-\frac{Rt}{L-M}}\right)$$

and $$\frac{dy}{dt} = 0 \quad \text{if} \quad \frac{R}{L-M}e^{-\frac{Rt}{L-M}} = \frac{R}{L+M}e^{-\frac{Rt}{L+M}}$$

Transposing the factors—

$$e^{-\frac{Rt}{L-M} + \frac{Rt}{L+M}} = \frac{L-M}{L+M}$$

$$e^{\frac{Rt(L-M-L-M)}{L^2-M^2}} = \frac{L-M}{L+M}$$

$$e^{-\frac{2MRt}{L^2-M^2}} = \frac{L-M}{L+M}$$

or $$e^{\frac{2MRt}{L^2-M^2}} = \frac{L+M}{L-M}$$

In order to find an expression for t, this equation must be changed to a log form, thus—

$$\log\left(\frac{L+M}{L-M}\right)=\frac{2MRt}{L^2-M^2}$$

or

$$t=\frac{L^2-M^2}{2MR}\log\left(\frac{L+M}{L-M}\right).$$

If three variables are concerned, say x, y and z, the relation between them being expressed by the equation $z=f(x, y)$, then in order to find the values of x and y for turning values of z, it is necessary to determine where the plane tangential to the surface is horizontal.

The algebraic problem is to find the values of x and y that satisfy simultaneously the equations $\frac{\partial z}{\partial x}=0$ and $\frac{\partial z}{\partial y}=0$, these derivatives being partial.

Example 12.—The electric time constant of a cylindrical coil of wire (*i. e.*, the time in which the current through the coil falls from its full value to a value equal to ·632 of this) can be expressed approximately by $K=\frac{mxyz}{ax+by+cz}$ where z is the axial length of the coil, y is the difference between the external and internal radii and x is the mean radius; a, b and c representing constants. If the volume of the coil is fixed, find the values of x and y which make the time constant as great as possible.

The volume V of the coil = cross section × length

i. e., $$V=2\pi x\times y\times z \text{ and } z=\frac{V}{2\pi xy}$$

$$K=m\left\{\frac{xyz}{ax+by+cz}\right\} \text{ and is a maximum}$$

when $$\frac{ax+by+cz}{xyz} \text{ or } \frac{a}{yz}+\frac{b}{xz}+\frac{c}{xy} \text{ is a minimum.}$$

Let
$$\begin{aligned}p&=\frac{a}{yz}+\frac{b}{xz}+\frac{c}{xy}\\&=\frac{a\times 2\pi xy}{yV}+\frac{b\times 2\pi xy}{xV}+\frac{c}{xy}\\&=\frac{2\pi xya}{yV}+\frac{2\pi xyb}{xV}+\frac{c}{xy}.\end{aligned}$$

Now
$$\begin{aligned}\frac{\partial p}{\partial x}&=\frac{2\pi a}{V}+\left(\frac{c}{y}\times-\frac{1}{x^2}\right)\\&=\frac{2\pi a}{V}-\frac{c}{x^2y}.\end{aligned}$$

Similarly $$\frac{\partial p}{\partial y} = \frac{2\pi b}{V} - \frac{c}{xy^2}.$$

Both $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ must be equated to zero,

so that $$\frac{2\pi a}{V} = \frac{c}{x^2 y}$$

i. e., $$x^2 y = \frac{cV}{2\pi a} \quad \ldots\ldots\ldots \quad (1)$$

and $$\frac{2\pi b}{V} = \frac{c}{xy^2}$$

i. e., $$xy^2 = \frac{cV}{2\pi b} \quad \ldots\ldots\ldots \quad (2)$$

To solve for x and y—

From (2)— $$x = \frac{cV}{2\pi b y^2}.$$

Substituting in (1)—
$$\frac{c^2V^2y}{4\pi^2b^2y^4} = \frac{cV}{2\pi a}$$

whence $$y^3 = \frac{cVa}{2\pi b^2}$$

or $$\underline{y = \sqrt[3]{\frac{caV}{2\pi b^2}}}$$

also $$\underline{x = \sqrt[3]{\frac{cbV}{2\pi a^2}}}.$$

Exercises 10.—On Maximum and Minimum Values.

1. If $M = 15x - \cdot 01x^2$, find the value of x that makes M a maximum.
2. Find the value of x that makes M a maximum if $M = 3\cdot 42x - \cdot 1x^2$.
3. M is a bending moment and x is a length; find x in terms of l M shall be a maximum, and find also the maximum value of M.
$$M = \frac{wx}{2}(l - x).$$
4. As for No. 3, but taking—
$$M = \frac{Wx}{3}\left(1 - \frac{x^2}{l^2}\right).$$
5. The work done by a series motor in time t is given by—
$$W = \frac{et(E - e)}{R}$$

where e = back E.M.F.
E = supply pressure
R = resistance of armature.

The electrical efficiency is $\frac{e}{E}$. Find the efficiency when the motor so runs that the greatest rate of doing useful work is reached.

In Nos. 6 to 8 find values of x which give turning values to y, stating the nature of these turning values.

6. $y = 4x^2+18x-41$. 7. $y = 5x-9x^2+18$.

8. $y = x^3+6x^2-15x+51$ (find also the value of x at the point of inflexion).

9. Sixteen electric cells, each of internal resistance 1 ohm and giving each 1 volt, are connected up in mixed circuit through a resistance of 4 ohms. Find the arrangement for the greatest current [say $\frac{16}{x}$ rows with x cells in each row].

10. If 40 sq. ft. of sheet metal are to be used in the construction of an open tank with square base, find the dimensions so that the capacity of the tank is a maximum.

11. Given that $W = 4C^2+\frac{74}{C}$, find a value of C that gives a turning value of W, and state the nature of this turning value.

12. M (a bending moment) $= W\frac{(l-x)}{l}(x+y)-Wy$. For what value of x is M a maximum? {W, l and y are constants.}

13. The cost C (in pounds sterling per mile) of an electric cable can be expressed by—

$$C = \frac{157}{x}+636x$$

where x is the cross section in sq. ins.

Find the cross section for which the cost is the minimum, and find also the minimum cost.

14. A window has the form of a rectangle together with a semicircle on one of its sides as diameter, and the perimeter is 30 ft. Find the dimensions so that the greatest amount of light may be admitted.

15. C, the cost per hour of a ship, in pounds, is given by—

$$C = 3{\cdot}2+\frac{s^3}{2200}$$

where s = speed in knots.

Find the value of s which makes the cost of a journey of 3000 nautical miles a minimum.

At speed 10% greater and less than this compare the total cost with its minimum value.

16. An isolated load W rolls over a suspension bridge stiffened with pin-jointed girders. When the load is at A, distant x from the centre, the bending moment at this section $= M_A = \frac{Wx}{2l^2}(l^2-4x^2)$. For what value of x is M_A a maximum?

17. A riveted steel tank of circular section open at the top has to be constructed to contain 5000 gals. of water. Find the dimensions so that the least possible amount of steel plate is required.

18. A canister having a square base is cut out of 128 sq. ins. of tin, the depth of the lid being 1 in. Find the dimensions in order that the capacity of the canister may be as large as possible.

19. The stiffness of a beam of rectangular section is proportional to the breadth and the cube of the depth. Find the ratio of the sides of the stiffest beam of rectangular section with a given perimeter

20. A load uniformly distributed over a length r rolls across a beam of length l, and the bending moment M due to this loading at a point is given by—

$$M = \frac{wry}{l}\left\{l - y + x - \frac{r}{2}\right\} - \frac{wx^2}{2}.$$

For what value of x is M a maximum?

21. Find the value of V (a velocity) that makes R (a resistance) a minimum when—

$$R = \frac{V^2}{54} - \frac{3(V-12)}{V+12}.$$

22. If $L = \sqrt{r^2 - x^2} - \frac{1}{r}(r^2 - x^2)$, find the value of x that makes L a maximum.

23. A jet of water, moving with velocity v, impinges on a plate moving in the direction of the jet with velocity u. The efficiency $\eta = \frac{2u(v-u)^2}{v^2}$. Find values of u for maximum and minimum efficiency, and find also the maximum efficiency.

24. If $\eta = \frac{2u(v-u)}{v^2}$, find the value of u for maximum value of η.

25. Given that $Q = K\mu T_1(\cos\theta - \sin\theta)$, find values of θ between $0°$ and $360°$ that make Q a maximum, treating K, μ and T_1 as constants.

26. A cylinder of a petrol engine is of diameter d and length l. Find the value of the ratio $\frac{d}{l}$ which makes $\frac{\text{area of exposed surface}}{\text{capacity}}$ a minimum. The volume must be treated as a constant.

27. If the exposed surface of a petrol engine cylinder is given by—

$S = 2\pi r^2 + 2\pi rl + \cdot 2r^2$, l being the length and r the radius,

find the value of the ratio $\frac{l}{r}$ that makes the ratio $\frac{\text{exposed surface}}{\text{capacity}}$ a minimum. The volume must be treated as a constant.

28. Given that $y = \frac{K^2 - 5\cdot 2K + 60\cdot 2}{K^2 + 9}$, find values of K for turning values of y.

29. If $M = \frac{wR^2}{2}\left(\frac{1}{4} - \sin^2\theta\right) - \cdot 934wR^2\left(\cos\theta - \sqrt{\frac{3}{2}}\right)$, for what values of θ is M a minimum? [M is the bending moment at a section of a circular arched rib loaded with a uniform load w per foot of span, and R is the radius of the arch.]

30. An open channel with side slopes at $45°$ is to have a cross section of 120 sq. ft. Determine the dimensions for the best section (*i. e.*, the section having the smallest perimeter for a given area).

31. If $M = \frac{w(2xl - 3x^2)^2}{8(l^2 - 2x^2)}$, find the value of x which makes M, a bending moment, a maximum. The final equation should be solved by plotting, a value being assumed for l.

32. In connection with retaining walls the following equation occurs—

$$P = \frac{\rho h^2}{72} \cdot \frac{1-\mu \tan \theta}{1-\mu^2+2\mu \cot \theta}.$$

Find an expression, giving the value of θ (in terms of $\tan \theta$), that makes P a maximum. $\{\mu$, ρ and h are constants.$\}$

33. Assuming that the H.P. of an engine can be expressed by the relation—

$$H = C(fnl^3 - K\rho n^3 l^5)$$

where C is a constant, l = stroke, p = pressure in piston rod due to the pressure on the piston, ρ = average density of the material of the engine, K = constant depending upon the mode of distribution of the mass of the engine parts, n = R.P.M., and f = safe stress in the material, find an expression for l giving the maximum H.P. for engines of different sizes.

34. Find the turning point of the probability curve—

$$y = \frac{1}{h\sqrt{\pi}} e^{-\frac{x^2}{h^2}}$$

and also the points of inflexion.

35. In a two-stage compressor, neglecting clearances, if P_1 and V_1 are the initial pressure and volume of the L.P. cylinder, P_2 the pressure in the intercooler, and P_3 the discharge pressure of the H.P. cylinder, the total work for the two cylinders is given by—

$$W = \frac{n}{n-1} P_1 V_1 \left\{ \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} + \left(\frac{P_3}{P_2}\right)^{\frac{n-1}{n}} - 2 \right\}.$$

For what value of P_2 is W a minimum, P_1, V_1, P_3 and n being regarded as constants?

36. Find the height h of a Warren girder to give the maximum stiffness, the stiffness being given by the expression—

$$\frac{f_c+f_t}{2Eh}\left\{l^2+\frac{ld}{4}+\frac{lh^2}{d}\right\}$$

d being the length of one bay and l the span, whilst f_c, f_t and E are constants for the material.

37. The efficiency of a reaction wheel may be expressed by—

$$\eta = \frac{2(n-1)}{(1+K)n^2-1}$$

For what value of n has η its maximum value?

38. The weight W of steam passing through an orifice, from pressure P_1 to pressure P_2, is given by—

$$W = A_2\sqrt{2g\frac{P_1}{v_1}\cdot\frac{n}{n-1}\left\{\left(\frac{P_2}{P_1}\right)^{\frac{2}{n}}-\left(\frac{P_2}{P_1}\right)^{\frac{n+1}{n}}\right\}}$$

If $n = 1{\cdot}135$, find the value of $\frac{P_2}{P_1}$ for which W is a maximum.

39. Find the height of the greatest cylinder that can be inscribed in the frustum of a paraboloid of revolution cut off by a plane perpendicular to the axis and distant 6 units from the origin. The paraboloid is generated by the revolution about the axis of x of the parabola $y^2 = 5x$.

40. If $M = W\left\{x - \frac{(x+y)^2}{l}\right\}$ where y and l are constants, find the value of x that makes M a maximum.

41. If T, t and T_f are the tight, slack and centrifugal tensions respectively in a belt passing round a pulley, and v = speed of the belt in feet per sec., then—

$$\text{H.P. transmitted} = \frac{v(T-t)}{550}.$$

Being given that $T_f = \frac{wv^2}{g}$, the maximum permissible tension in the belt $= T_m = T + T_f$, μ = coefficient of friction between belt and pulley, θ = angle of lap of belt in radians, and $\frac{T}{t} = e^{\mu\theta}$, find the value of T_f in terms of T_m so that the maximum H.P. is transmitted.

42. If $y = 3x^4 + 2x^3 - 78x^2 - 240x + 54$, find the values of x which give turning values to y, stating the nature of these turning values; and find also the values of x at the points of inflexion.

43. The radial stress in a rotating disc

$$\begin{aligned} &= p_x \\ &= \frac{w\omega^2}{8gm}(3m+1)\left(R_1^2 + R_2^2 - \frac{R_1^2R_2^2}{x^2} - x^2\right) \end{aligned}$$

in which expression x is the only variable.

Find the value of x which gives to p_x its maximum value, and state this value of p_x.

44. A pipe of length l and diameter D has at one end a nozzle of diameter d through which water is discharged from a reservoir, the level of the water in which is maintained at a constant head h above the centre of the nozzle. Find the diameter of the nozzle so that the kinetic energy of the jet may be a maximum; the kinetic energy being expressed by—

$$\rho\frac{\pi}{4}d^2\left(\frac{2gD^5h}{D^5 + 4fld^4}\right)^{\frac{3}{2}}$$

[*Hint.*—If K = kinetic energy, write—

$$K = \frac{\rho\pi}{4}\left(\frac{2gD^5hd^{\frac{4}{3}}}{D^5 + 4fld^4}\right)^{\frac{3}{2}}$$

and find the value of d for the maximum value of $K^{\frac{2}{3}}$.]

45. Prove that the cuboid of greatest volume which can be inscribed in a sphere of radius a is a cube of side $\cdot 577a$.

46. The velocity of the piston of a reciprocating engine can be expressed by—

$$2\pi nr\left(\frac{\sin 2\theta}{2m} + \sin\theta\right)$$

where θ is the inclination of the crank to the line of stroke.

If $m = \frac{\text{connecting-rod length}}{\text{length of crank}} = 8$, find the values of θ between 0° and 360° that make the velocity a maximum.

Calculation of Small Corrections.—Differentiation finds another application in the calculation of small corrections.

Thus an experiment might be carried out, certain readings being taken, and results deduced from these readings; then if there is a possibility of some slight error in the readings and it is required to find the consequent error in the calculated result, we may proceed to find that error in the manner now to be explained.

Suppose we have two quantities A and B connected with one another by a formula $A = KB$; then if the value of B is slightly inaccurate the error in A will depend on this error in B, and also on the rate at which A changes with regard to B. *E. g.*, if A changes three times as fast as B and the error in B is $\cdot 1\%$, then the consequent error in A must be $3 \times \cdot 1$ or $\cdot 3\%$.

We might also look upon this question from a different point of view. Suppose that a reading, instead of being x, as it should have been, was slightly larger, say $x + \delta x$, *i. e.*, the measured value of x would be represented by OB and not OA (Fig. 26), then the error is δx or $\frac{\delta x}{x} \times 100\%$. This error causes an error in the value of y, so that the calculated value of y is BQ and not AP, *i. e.*, the error is δy.

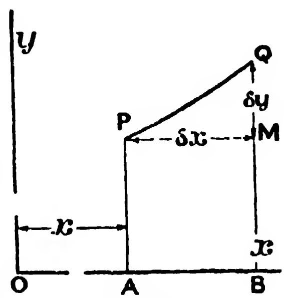


FIG. 26.

To compare these errors we proceed as follows: $\frac{\delta y}{\delta x}$ = the slope of the chord PQ, and if δx is very small (as it should be, for otherwise the experiment would be repeated), then this would also be the slope of the tangent at both P and Q, or, approximately—

$$\frac{\delta y}{\delta x} = \frac{dy}{dx}$$

$$i.\ e., \qquad \delta y = \frac{dy}{dx} \times \delta x$$

or, error in y = rate at which y changes with regard to x $\times$ error in x.

Example 13.—In the measurement of the diameter of a shaft, of which the actual diameter was 4 ins., an error of 2% was made; what was the consequent error in the weight?

Here—
$$W = \frac{\pi}{4}d^2 l\rho, \text{ where } \rho \text{ is the density}$$
$$= Kd^2, \quad \text{where } K = \frac{\pi}{4}l\rho.$$

Now the error in the diameter $= \delta d = \frac{2}{100} \times 4 = \cdot 08$ in.

also
$$\frac{d.W}{d.d} = \frac{d.Kd^2}{d.d} = 2Kd$$
$\therefore$
$$\delta W = 2Kd \times \cdot 08$$
or the percentage error
$$= \frac{\delta W}{W} \times 100 = \frac{2Kd \times \cdot 08}{Kd^2} \times 100$$
$$= \frac{\cdot 16}{d} \times 100$$
$$= \frac{16}{4} = 4\%.$$

Example 14.—If some torsion experiments are being made on shafts varying in diameter from 1 in. to 5 ins.; then, allowing a maximum error of ·5% in the measurement of the diameters, what is the range of the errors in the stress? Given that $T = \frac{\pi}{16} f d^3$.

$$\text{The stress } f = \frac{16T}{\pi} \times \frac{1}{d^3}$$
hence
$$\frac{df}{d\,d} = \frac{16T}{\pi} \times \left(-\frac{3}{d^4}\right) = -\frac{48T}{\pi d^4}.$$

Now the error in the diameter $= \delta.d$ is ·5%

i. e.,
$$\delta d = \frac{\cdot 5}{100} \times d.$$

Hence the error in $f = \frac{df}{d.d} \times \delta d = -\frac{48T}{\pi d^4} \times \frac{\cdot 5d}{100}$

i. e., the percentage error in f—
$$= 100 \times \frac{\delta f}{f} = 100 \times -\frac{48T}{\pi d^4} \times \frac{\cdot 5d}{100} \times \frac{\pi d^3}{16T}$$
$$= -1 \cdot 5.$$

$$\left.\begin{array}{l}\text{Thus the smallest error} = \cdot 015 \times \text{smallest stress} \\ \quad \text{and the largest error} = \cdot 015 \times \text{largest stress}\end{array}\right\}.$$

If the error in the measurement of the diameter is on the high side, then the stress, as calculated, will be too low.

Expansion of Functions in Series. Theorems of Taylor and Maclaurin.—Many of the simpler functions, such as $\log_e (1+x)$, $\sin x$, $\cos x$, etc., can be expressed as the sums of series. These functions can be expressed in terms of these series by the use of a theorem known as Maclaurin's.

Let $f(x)$ stand for the function of x considered, and let $f(x) = a+bx+cx^2+dx^3+ \ldots$, to be true for all values of x,

$$i.\,e.,\quad f(0) = a+(b\times 0)+(c\times 0^2)+(d\times 0^3)+ \ldots$$
$$= a.$$

We assume that the differentiation of the right-hand side term by term gives the derivative of $f(x)$.

Differentiate both sides with regard to x.

$$\text{Then—}\qquad \frac{df(x)}{dx} \text{ or } f'(x) = b+2cx+3dx^2+ \ldots$$

This must be true for all values of x; thus, let $x = 0$

$$\text{then—}\qquad f'(x) \text{ when } x = 0 \text{ or } f'(0) = b$$

$f'(0)$ implying that $f'(x)$ or $\dfrac{df(x)}{dx}$ is first found and then the value 0 substituted for x throughout.

$$\text{Differentiating again, } \frac{d^2f(x)}{dx^2} \text{ or } f''(x) = 2c+6dx+ \ldots$$

$$\text{and}\qquad f''(0) = 2c$$

$$i.\,e.,\qquad c = \frac{f''(0)}{2}.$$

$$\text{Similarly,}\qquad f'''(x) = 6d + \text{terms containing } x \text{ and higher powers of } x,$$

$$\text{whence}\qquad f'''(0) = 6d \quad\text{or}\quad 1.2.3.d.$$

$$\therefore\qquad d = \frac{f'''(0)}{1.2.3} \quad\text{or}\quad \frac{f'''(0)}{\underline{\lfloor 3}}$$

Accordingly we may write the expansions—

$$f(x) = f(0)+f'(0)x+\frac{f''(0)x^2}{\underline{\lfloor 2}}+\frac{f'''(0)x^3}{\underline{\lfloor 3}}+ \ldots$$

This is *Maclaurin's Theorem.* By a similar investigation we might obtain Taylor's Theorem, which may be regarded as a more general expression of the foregoing.

Taylor's Theorem. In this the expansion is of $f(x+h)$ and not $f(x)$; thus—

$$f(x+h) = f(h)+xf'(h)+\frac{x^2}{\underline{\lfloor 2}}f''(h)+\frac{x^3}{\underline{\lfloor 3}}f'''(h)+ \ldots$$

or, as it is sometimes written, to give an expansion for $f(x)$—

$$f(x) = f(h)+(x-h)f'(h)+\frac{(x-h)^2}{\underline{\lfloor 2}}f''(h)+\frac{(x-h)^3}{\underline{\lfloor 3}}f'''(h)+ \ldots$$

If in either of these two expansions we make $h = 0$, then Maclaurin's series results.

We may now utilise these theorems to obtain series of great importance.

Example 15.—To find a series for $\cos x$.

$$\begin{aligned}
&\text{Let} && f(x) = \cos x \\
&\quad\text{then} && f(0) = \cos 0 = 1. \\
&\text{Also}\quad f'(x),\ i.e.,\ \frac{d\cos x}{dx} && = -\sin x \\
&\quad\text{so that} && f'(0) = -\sin 0 = 0. \\
&\text{Again} && f''(x) = \frac{d}{dx}(-\sin x) = -\cos x \\
&\quad\text{so that} && f''(0) = -\cos 0 = -1 \\
&\quad\text{and} && f'''(x) = \frac{d}{dx}(-\cos x) = \sin x \\
&\quad\text{so that} && f'''(0) = \sin 0 = 0. \\
&\text{Now} && f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{\lfloor 2} + \ldots; \\
&\text{Therefore} && \cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots
\end{aligned}$$

Example 16.—To find a series for $\log_e(1+x)$.

$$\begin{aligned}
&\text{Let} && f(x) = \log_e(1+x) \\
&\quad\text{then} && f(0) = \log 1 = 0. \\
&\text{Now} && f'(x) = \frac{d\log(1+x)}{dx} = \frac{1}{1+x} \\
&\quad\text{so that} && f'(0) = \frac{1}{1} = 1 \\
& && f''(x) = \frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2} \\
&\quad\text{so that} && f''(0) = \frac{-1}{1} = -1 \\
&\quad\text{and} && f'''(x) = \frac{d}{dx}\left(\frac{-1}{(1+x)^2}\right) = \frac{2}{(1+x)^3} \\
&\quad\text{so that} && f'''(0) = \frac{2}{1} = 2. \\
&\text{Hence} && \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots;
\end{aligned}$$

(Compare with the series found by an entirely different method in Part I, p. 470.)

Example 17.—Prove that $e^{jx} = \cos x + j\sin x$, where $j = \sqrt{-1}$. This equation is of great importance, since it links up the exponential and the trigonometric functions.

To find a series for $\sin x$.

$$
\begin{array}{llll}
\text{Let} & f(x) = \sin x & \text{then} & f(0) = \sin 0 = 0 \\
 & f'(x) = \cos x & & f'(0) = \cos 0 = 1 \\
 & f''(x) = -\sin x & & f''(0) = -\sin 0 = -0 \\
 & f'''(x) = -\cos x & & f'''(0) = -\cos 0 = -1.
\end{array}
$$

Hence $$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \ldots$$

and $$j \sin x = j\left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \ldots\right)$$

To find a series for e^{jx}.

$$
\begin{array}{lll}
\text{Let} & f(x) = e^{jx} & f(0) = e^0 = 1 \\
 & f'(x) = je^{jx} & f'(0) = je^0 = j = \sqrt{-1} \\
 & f''(x) = j^2e^{jx} & f''(0) = j^2e^0 = j^2 = -1 \\
 & f'''(x) = j^3e^{jx} & f'''(0) = j^3e^0 = j^3 = -\sqrt{-1}.
\end{array}
$$

Hence $$e^{jx} = 1 + jx + \frac{j^2x^2}{\underline{|2}} + \frac{j^3x^3}{\underline{|3}} \ldots$$

Now $\cos x + j \sin x$ (the series for $\cos x$ having been found in *Example* 15).

$$
\begin{aligned}
&= \left(1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \ldots\right) + j\left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \ldots\right) \\
&= 1 + jx - \frac{x^2}{\underline{|2}} - \frac{jx^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} - \ldots \\
&= 1 + jx + \frac{j^2x^2}{\underline{|2}} + \frac{j^3x^3}{\underline{|3}} + \frac{j^4x^4}{\underline{|4}} + \ldots \qquad \left\{\begin{array}{l} \text{For } j^2 = -1 \\ \phantom{\text{For }} j^3 = -\sqrt{-1} \\ \phantom{\text{For }} j^4 = +1, \text{ etc.} \end{array}\right\} \\
&= e^{jx}.
\end{aligned}
$$

Use might be made of Taylor's Theorem to determine a more correct solution to an equation when an approximate solution is known; for, taking the first two terms of the expansion only—

$$f(x+h) = f(h) + xf'(h)$$

or interchanging x and h, as a matter of convenience, then—

$$f(h+x) = f(x) + hf'(x).$$

If h is small compared with x, the assumption that two terms of the series may be taken to represent the expansion is very nearly true.

Suppose that a rough approximation for the root has been found (by trial and error); denote this by x. Let the true solution be $x+h$; then by substitution in the above equation the value of h can be found, and thence that of $x+h$.

As an illustration, consider the following case: A rough test gives 2·4 as a solution of the equation $x^4-1{\cdot}5x^3+3{\cdot}7x=21{\cdot}554$.
It is required to find a solution more correct.

Here $x=2{\cdot}4$ and $f(x)=x^4-1{\cdot}5x^3+3{\cdot}7x-21{\cdot}554$
so that $f(2{\cdot}4)=33{\cdot}178-20{\cdot}736+8{\cdot}88-21{\cdot}554=-{\cdot}232$.

If the correct value of h is found, then $f(x+h)$ must $=0$.

Hence— $f(x+h)=f(x)+hf'(x)$
i. e., $0=-{\cdot}232+hf'(2{\cdot}4)$.
[Now— $f(x)=x^4-1{\cdot}5x^3+3{\cdot}7x-21{\cdot}554$
$\therefore$ $f'(x)=4x^3-4{\cdot}5x^2+3{\cdot}7$
so that $f'(2{\cdot}4)=55{\cdot}30-25{\cdot}92+3{\cdot}7=33{\cdot}08$.]
Hence $0=-{\cdot}232+(h\times 33{\cdot}08)$
or $h=\dfrac{{\cdot}232}{33{\cdot}08}={\cdot}007$.

Hence a more correct approximation is $2{\cdot}4+{\cdot}007$
i. e., $x=2{\cdot}407$ is the solution of the equation.

This method may thus be usefully employed in lieu of the graphic method when extremely accurate results are desired.

The following example illustrates the process of interpolation necessary in many cases where the tables of values supplied are not sufficiently detailed for the purpose in hand; and in view of the importance of the method, every step in the argument should be thoroughly understood.

Example 18.—It is desired to use some steam tables giving the pressures for each 10° difference of temperature, to obtain the accurate value of $\frac{dp}{dt}$ when $t=132°$ C. The figures in the line commencing with $t=130°$ C. (the nearest to 132°) are as follows:—

t	p	$\frac{dp}{dt}$	$\frac{d^2p}{dt^2}$	$\frac{d^3p}{dp^3}$	$\frac{d^4p}{dt^4}$
130	2025·717	60·5995	1·47051	·026392	·0002738

Calculate, very exactly, the value of $\frac{dp}{dt}$ when $t=132°$ C.

Taylor's theorem may here be usefully employed, using the form—

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\underline{|2}}f''(x)+\frac{h^3}{\underline{|3}}f'''(x)+\frac{h^4}{\underline{|4}}f''''(x)+\ \ldots$$

Let $\qquad f(x) = \frac{dp}{dt}$ when $t = 130$

and $f(x+h) = \frac{dp}{dt}$ when $t = 132$, so that $h = 2$

Then $\qquad f'(x) = \frac{d}{dt}\left(\frac{dp}{dt}\right) = \frac{d^2p}{dt^2}$ and $f''(x) = \frac{d^3p}{dt^3}$, etc., t having the value 130.

Thus the expansion may be re-written as—

$$\left(\frac{dp}{dt}\right)_{132} = \left(\frac{dp}{dt}\right)_{130} + 2\left(\frac{d^2p}{dt^2}\right)_{130} + \frac{2^2}{\underline{|2}}\left(\frac{d^3p}{dt^3}\right)_{130} + \frac{2^3}{\underline{|3}}\left(\frac{d^4p}{dt^4}\right)_{130}$$

and substituting the values from the table—

$$\left(\frac{dp}{dt}\right)_{132} = 60{\cdot}5995 + (2 \times 1{\cdot}47051) + (2 \times {\cdot}026392) + \left(\frac{4}{3} \times {\cdot}0002738\right)$$
$$= \underline{63{\cdot}59367.}$$

Exercises 11.—On the Calculation of Small Corrections and Expansion in Series.

1. If $R = R_0(1 + at + bt^2)$ when R_0 (the resistance of a conductor at 0° C.) is 1·6, a (the temperature-resistance coefficient of the material) $= {\cdot}00388$ and $b = {\cdot}000000587$, find the error in R (the resistance at temperature t° C.) if t is measured as 101 instead of 100.

2. The quantity Q of water flowing over a notch is given by $Q = \frac{8}{15} \times {\cdot}64 \times \sqrt{2g}\,.\,H^{\frac{5}{2}}$, where H is the head at the notch. What is the percentage error in Q caused by measuring H as ·198 instead of ·2 ?

3. If $y = 4x^{1{\cdot}76}$, $y = 17{\cdot}33$ when $x = 2{\cdot}3$. What will be the change in y consequent on a change of x to 2·302 ?

4. A rough approximation gives $x = -2{\cdot}44$ as a solution of the equation $10^{\frac{2x}{3}} = 16 + 4x - x^2$. Find a more correct root.

5. Determine the value of x to satisfy the equation $x^{1{\cdot}8} - 3 \sin x = 3$, having given that it is in the neighbourhood of 2·67.

6. The height h of a Porter governor is expressed by—

$$h = \frac{W+w+f}{w} \cdot \frac{g}{n^2}$$

where n is the number of revolutions per minute. If W = 100, $w = 2$ and $f = 10$, find the change in the height due to a change in the speed from 200 to 197 r.p.m.

7. In calculating the co-ordinates of a station in a survey it was thought that there was a possibility of an error of 3 minutes (*i. e.*, $1\frac{1}{2}$ either way) in the reading of the bearing. If the bearing of a line was read as 7° 12′ and the length of the line was 2 chains 74 links,

find the possible errors in the co-ordinates of the distant end of the line. [Co-ordinates are length $\times$ cos (bearing) and length $\times$ sine (bearing).]

8. Find by the methods of this chapter a series for a^x.

9. Using the figures given in Example 18, p. 112, calculate very exactly the pressure p at 133° C.

10. The equation $d^3+\cdot 65d-\cdot 5=0$ occurred when finding the sag of a cable. A rough plotting gives the solution to be in the neighbourhood of $\cdot 53$: find a more exact root.

11. One root of the equation $6a^2-11a-34\cdot 8=0$ is in the neighbourhood of $3\cdot 5$. Determine its value more exactly.

12. In finding r, the ratio of expansion for a certain single cylinder engine the equation $1+\log_e r-\cdot 389r=0$ was obtained. By plotting, a value of r is found, viz. $7\cdot 88$, which apparently satisfies the equation. Find the more exact root.

Find solutions more nearly correct in Nos. 13 and 14, using the approximate roots stated.

13. $e^{3x}-5x^2-17=0:1\cdot 03$.

14. $d^3-4\cdot 19d-\cdot 35=0:1\cdot 34$.

CHAPTER V

INTEGRATION

HAVING discussed the section of the Calculus which treats of differentiation, we can now proceed to the study of the process of integration, this having a far more extensive application, and being, without doubt, far more difficult to comprehend.

As with the differentiation, it is impossible fully to appreciate this branch of the subject unless much careful thought is given to the fundamental principles; and accordingly the introduction to the Integral Calculus is here treated at great length, but in a manner which, it is hoped, will commend itself.

Meaning of Integration.—The terms *integer* and *integral* convey the idea of totality; an integer being, as we know, a whole number, and thus the sum of its constituent parts or fractions. The process of *integration* in the same way implies a summation or a totalling, whereas that of differentiation is the determination of rates of change or the comparison of small differences. Differentiation suggests subtraction or differencing, whilst integration suggests addition; differentiation deals with rates of change, integration with the results of the total change; differentiation involves the determination of slopes of curves, and integration the determination of areas of figures. Integration is, in fact, the converse to differentiation, and being therefore a *converse* operation is essentially more difficult to perform. [As instances of this statement contrast the squaring a quantity with the extraction of a square root, or the removal of brackets with factorisation.]

A converse operation is rather more vague as concerns the results than a direct; for when performing a direct operation one result only is obtainable, but the results of a converse operation may be many, as we shall find, for example, when dealing with indefinite integrals.

To illustrate the connection between differentiation and integration, consider the familiar case of velocity and acceleration. Suppose values of v and t are given, as in the table :—

t	·1	·15	·20	·25	·30
v	28·4	29·7	30·5	33·4	36·5

Then

δv	1·3	·8	2·9	3·1
δt	·05	·05	·05	·05
$a = \dfrac{\delta v}{\delta t}$	26	16	58	62

The accelerations are here found by comparing differences of velocity with differences of time.

Regard the question from the other point of view: assume that these accelerations are given and we wish to determine the total change in the velocity in the given period of time. The total change must be given by the sum of the changes in the small periods of time; in the first period of ·05 sec. the average acceleration was 26, *i. e.*, the velocity was being increased at the rate of 26 units per sec. each sec.; and therefore the change in the velocity in ·05 sec. $= 26 \times {\cdot}05$ units per sec.

$= 1{\cdot}3$ units per sec.

In the successive periods the changes in velocity are ·8, 2·9 and 3·1 respectively.

Hence the total change in the velocity over the period ·2 sec. $= 1{\cdot}3 + {\cdot}8 + 2{\cdot}9 + 3{\cdot}1 = 8{\cdot}1$ units per sec., or if the initial velocity was 28·4, the final velocity was $28{\cdot}4 + 8{\cdot}1 = 36{\cdot}5$. Note that the acceleration is given by the fraction $\dfrac{\delta v}{\delta t}$, whilst a small change in the velocity is of the nature $a\delta t$, or the total change of velocity = sum of all small changes $= \Sigma a\delta t$.

We can thus find integrals by working through the processes of differentiation, but in the reverse order. If a function, expressed in terms of symbols, has to be integrated, it is an advantage to transform the rules for differentiation into forms more readily applicable; the method, however, being entirely algebraic.

If numerical values alone are given, the integration resolves itself into a determination of an area.

Hence—

Considered from an *algebraic standpoint*—

Differentiation implies the calculation of rates of change;

Integration implies the summation of small quantities.

From the *graphic standpoint*—

Differentiation is concerned with the measurement of slopes of curves;

Integration is concerned with the measurement of areas under curves.

Just as special symbols are used to denote the processes of differentiation, so also there are special symbols for expressing the processes of integrations.

Regarding an integral as an area, it must be of two dimensions, a length and a breadth; and we have seen in an earlier chapter (Part I, Chap. VII) that in order to ascertain an area correctly its base must be divided up into small elements, the smaller the better, these elements not necessarily being of the same length, but all being small. Thus, to find the area ABCD (Fig. 27) we can suppose it divided up into small strips, as EFGH, then find the area of each of these and add the results. The portion EH of the curve is very nearly straight, so that EFGH is a trapezoid, and hence its area = mean height × width. Now its mean height, FE and GH are practically the same, so that any one of them can be denoted by y; also the width FG of the strip is a small element of the base, *i. e.*, is δx.

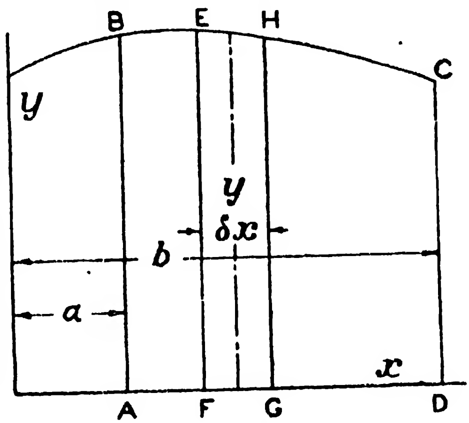


FIG. 27.

Hence, the area of the strip EFGH $= y \times \delta x$, and the total area between the curve, the bounding ordinates and the axis of x must equal the sum of all products like $y\delta x$, or, as it might be expressed—

$$\text{Area} = \Sigma y\delta x \text{ (approximately)}.$$

However small the width of the strips are made, this sum only gives the area approximately, but as δx is diminished the result approaches the true more and more closely.

Therefore, bearing in mind our previous work on limits, we can say that the limiting value of $\Sigma y\delta x$ must give the area exactly. To this limiting value of the sum different forms of symbols are

attached, the Σ and δ being replaced by the English forms $\int$ and d respectively, so that the area between the curve and the axis of $x = \int y dx$. There is no limit placed to this area in any horizontal direction, so that the area is not defined by the given formula.

Hence $\int y dx$ is spoken of as an *indefinite integral.*

The x is again the I.V., and the size of the area will depend on the values given to it. Suppose that when $y = AB$, $x = a$, and when $y = CD$, $x = b$; then the range of x is from a to b if it is the area ABCD that is considered. Accordingly we can state that the area $ABCD = \int y dx$, the value of this integral being found between $x = a$ and $x = b$, or, as it is written for brevity, $\int_{x=a}^{x=b} y dx$, or, more shortly still, $\int_a^b y dx$, it being clearly understood that the limits a and b apply to the I.V., *i.e.*, that quantity directly associated with the "d."

It is evident that ABCD is a definite area, having one value only, and thus $\int_a^b y dx$ is termed a *definite integral.*

The most convenient method for determining areas (provided that a planimeter is not handy) is undoubtedly the "sum curve" method treated in Part I, Chap. VII; the great virtue of it being that the growth of the area is seen, and that either any portion or the whole of the area of the figure can be readily found by reading a particular ordinate.

In view of the great usefulness of the process of integration by graphic means, the method is here explained in detail, following exactly the plan adopted in Part I, Chap. VII.

Graphic Integration is a means of summing an area with the aid of tee and set square, by a combination of the principles of the "addition of strips" and "similar figures." An area in Fig. 28 is bounded by a curve $a'b'z'$, a base line az and two vertical ordinates aa' and zz'. The base is first divided in such a way that the widths of the strips are taken to suit the changes of curvature between a' and z', and are therefore not necessarily equal; and mid-ordinates (shown dotted) are erected for every division. Next the tops of the mid-ordinates are projected horizontally on to a vertical line, as BB'. A *pole* P is now chosen to the left of that vertical; its distance from it, called the polar distance p, being a round number of horizontal units. The pole is next joined to each of the projections in turn and parallels are drawn across the

corresponding strips so that a continuous curve results, known as the **Sum Curve.** Thus *am* parallel to PB′ is drawn from *a* across the first strip; *mn* parallel to PC′ is drawn from *m* across the second strip, and so on.

The ordinate to the sum curve through any point in the base gives the area under the original or *primitive* curve from *a* up to the point considered.

Referring to Fig. 28—

$$\text{Area of strip } abb'a' = ab \times \text{AB}$$

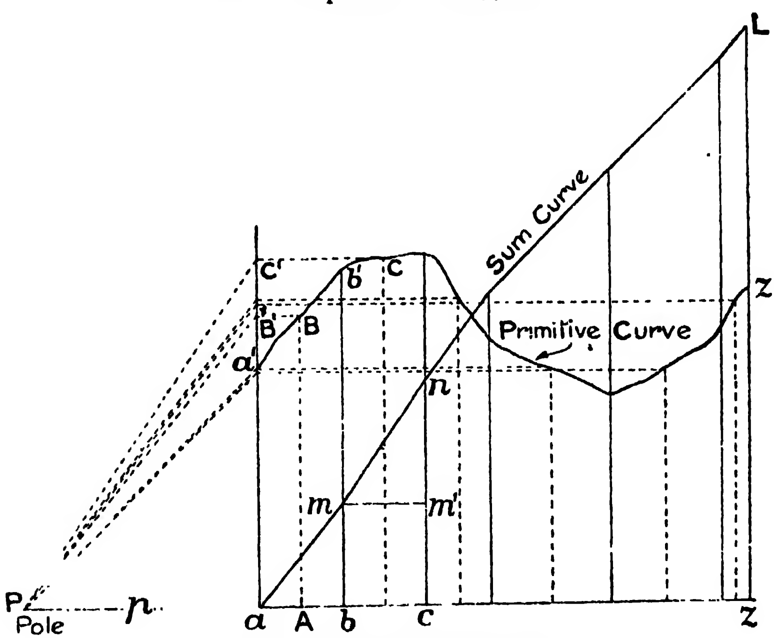


FIG. 28.—Graphic Integration.

but, by similar figures—

$$\frac{\text{B}'a \text{ or BA}}{p} = \frac{bm}{ab}$$

whence $\text{AB} \times ab = p \times bm$

i. e., $bm = \dfrac{\text{area of strip}}{p}$ or area of strip $= p \times bm$

i. e., bm measures the area of the first strip to a particular scale, which depends entirely on the value of p.

In the same way $nm' \quad \dfrac{\text{area of second strip}}{p}$

and by the construction nm' and bm are added, so that—

$$cn = \frac{\text{area of 1st and 2nd strips}}{p}$$

or— area of 1st and 2nd strips $= p \times cn$

Thus, summing for the whole area—

$$\text{Area of } aa'z'z = p \times z\text{L}.$$

Thus the scale of area is the old vertical scale multiplied by the polar distance; and accordingly the polar distance should be selected in terms of a number convenient for multiplication.

E. g., if the original scales are—

$$1'' = 40 \text{ units vertically}$$
$$\text{and} \quad 1'' = 25 \text{ units horizontally}$$

and the polar distance is taken as $2''$, *i. e.*, 50 horizontal units; then the new vertical scale—

$$= \text{old vertical scale} \times \text{polar distance}$$
$$= 40 \times 50 = 2000 \text{ units per inch.}$$

If the original scales are given and a *particular scale* is desired for the sum curve, then the polar distance must be *calculated* as follows—

$$\text{Polar distance in horizontal units} = \frac{\text{new vertical scale}}{\text{old vertical scale}}$$

E. g., if the primitive curve is a "velocity-time" curve plotted to the scales, $1'' = 5$ ft. per sec. (vertically) and $1'' = \cdot 1$ sec. (horizontally), and the scale of the sum curve, which is a "displacement-time" curve, is required to be $1'' = 2{\cdot}5$ ft., then—

$$\text{Polar distance (in horizontal units)} = \frac{2{\cdot}5}{5} = \cdot 5$$

and since $1'' = \cdot 1$ unit along the horizontal, the polar distance must be made $5''$.

Integration is not limited to the determination of areas only; true, an integral may be regarded as an area, but if the ordinate does not represent a mere length, but, say, an area of cross section, the value of the integral will in such cases measure the volume of the solid.

Our standard form throughout will be for the area of the figure as plotted on the paper, viz., $\int y dx$, where y is an ordinate and δx an element of the base, but y and x may represent many different quantities.

Thus, suppose a curve is plotted to represent the expansion of

a gas; if, as is usual, pressures are plotted vertically and volumes horizontally, the ordinate is p and an element of the base is δv; hence the area under the curve $= \int_{v_1}^{v_2} p\,dv$ (if the initial and final volumes are v_1 and v_2 respectively), and since this is of the nature pressure $\times$ volume, *i. e.*, $\frac{lbs}{(ft)^2} \times (ft)^3$ or ft. lbs., the area must represent the work done in the expansion.

To illustrate such a case:—

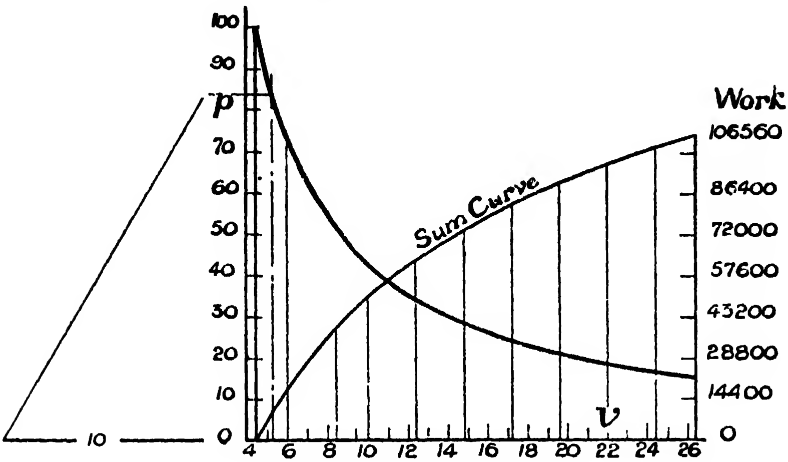


FIG. 29.—Expansion of Steam.

Example 1.—It is required to find the work done in the expansion of 1 lb. of dry saturated steam from pressure 100 lbs. per sq. in. to pressure 15 lbs. per sq. in.

From the steam tables the following corresponding values of p and v are found:—

v (cu. ft. per lb.)	4·44	5·48	7·16	10·50	13·72	20	26·4
p (lbs. per sq. in.)	100	80	60	40	30	20	15

By plotting these values, p vertically, the expansion curve is obtained (Fig. 29); this being the primitive curve.

Selecting a polar distance equivalent to 10 horizontal units, we proceed to construct the sum curve, the last ordinate of which measures to a certain scale the work done in the expansion. Now the new vertical scale = old vertical $\times$ 10, since the polar distance = 10; and also we must multiply by 144, since the pressures are expressed

in lbs. per sq. in. and must be converted to lbs. per sq. ft., so that the work done may be measured in ft. lbs.

According to this modified scale the last ordinate is read off as 106560; thus the work done = 106560 ft. lbs.

or, as it would be written in more mathematical language—

$$\int_{4\cdot 44}^{26\cdot 4} p\,dv = 106560.$$

Example 2.—The diameters of a tapering stone column, 20 ft. long, at 6 equidistant places were measured as 2·52, 2·06, 1·54, 1·15, ·80 and ·58 ft. respectively.

Find its weight at 140 lbs. per cu. ft.

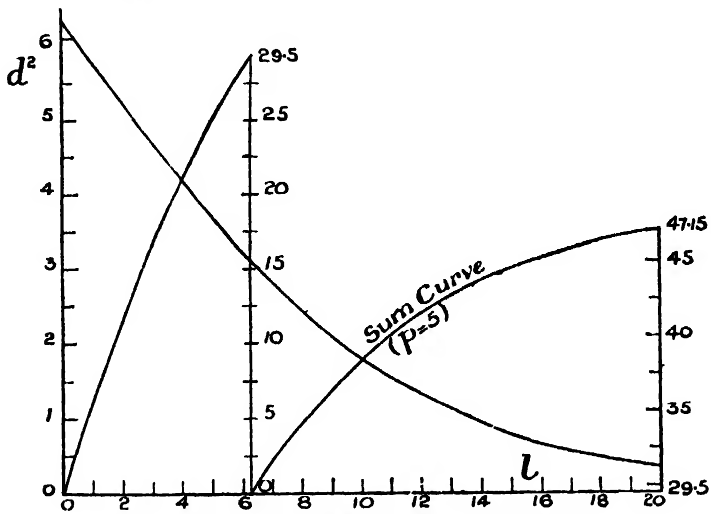


FIG. 30.—Problem on Stone Column.

The volume will be obtained by plotting the areas against the length and summing. Now the area of any section $= \frac{\pi}{4}d^2$, and the total volume will be the sum of the volumes of the small elements into which the solid may be supposed to be divided.

$$\text{Thus the volume} \quad = \int_0^{20} \mathrm{A}\,dl = \int_0^{20} \frac{\pi}{4} d^2.dl.$$

$$\text{and the weight} \quad = 140\int_0^{20} \frac{\pi}{4} d^2.dl.$$

Since $\frac{\pi}{4}$ is a constant multiplier, it can be omitted until the end, for its effect is simply to alter the final scale; **hence a constant factor before integration remains so after.**

$$\text{Hence the weight} = \frac{140 \times \pi}{4}\int_0^{20} d^2.dl = 110\int_0^{20} d^2.dl.$$

The integral will be of the standard form if for d^2 we write y and if for l we write x, so that we see that ordinates must represent d^2 and abscissæ lengths, and hence the table for plotting reads :—

l	0	4	8	12	16	20
y or d^2	6·35	4·24	2·37	1·32	·64	·336

Plotting these values and thence constructing the sum curve (see Fig. 30), we find the last ordinate to be 47·15, and this is the value of $\int_0^{20} d^2.dl$.

$$\therefore \qquad \text{Weight} = 110\int_0^{20} d^2.dl = 110 \times 47{\cdot}15 = \underline{5187 \text{ lbs.}}$$

Application of Integration to "Beam" Problems.—At an earlier stage (see p. 38) it has been demonstrated that the shear at any point in the length of a beam loaded in any way whatever is given by the rate of change of the bending moment in the neighbourhood considered, this being the *space rate* of change. Conversely, then, the bending moment must be found by summating the shearing force; and hence, if the shear curve is given, its sum curve is the curve of bending moment.

In the majority of problems the system of loading is given, from which the curve of loads can be drawn. Then, since the shear at any section is the sum of all the forces to the right or left of that section, the sum curve of the load curve must be the shear curve; continuing the process, the sum curve of the shear curve, *i. e.*, the second sum curve from the load curve as primitive, is the curve of bending moment and the fourth sum curve is the deflected form.

Expressing these results or statements in the notation of the calculus; L, S and M being the respective abbreviations for loading, shear and bending moment—

$$S = \int L dx$$
$$M = \int S dx = \int\left(\int L dx\right)dx = \iint L(dx)^2$$

[$\iint L(dx)^2$ being termed a *double integral*]

and the deflection $y = \iint M(dx)^2$ or $\iiiint L(dx)^4$.

If the loading is not uniform, but continuous, the summation must be performed graphically. [The link polygon method largely used obviates half these curves, *e. g.*, the link polygon for the loads gives at once the curve of bending moment.]

Example 3.—The loading on a beam, 24 ft. long, simply supported at its ends varies continuously, as shown in the table. Draw diagrams of shearing force and bending moment, stating clearly the maximum values of the shearing force and the bending moment.

Distance from one end (ft.)	0	4	7	10	12	14	17	20	24
Load in tons per ft.	·44	·58	·86	1·06	1·1	1·06	·86	·58	·44

The curve of loads is first plotted, as in Fig. 31.

By sum-curving this curve, we obtain the curve of shearing force, although no measurements can be made to it until account has been taken of the support reactions.

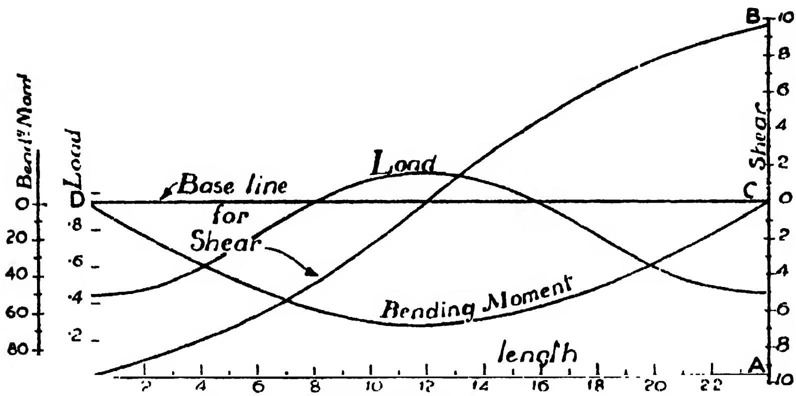


FIG. 31.—Problem on Loaded Beam.

To find the reactions at the ends: We know that these must be equal, since the loading is symmetrical, each reaction being one-half of the total load. Now the last ordinate AB of the sum curve of the load curve is 19; thus the reactions are each 9·5. Bisecting AB, or, in other words, marking off a length AC to represent the reaction at A, we draw a horizontal, and this is the true base line for the curve of shear; any ordinate to the curve of shear from this base giving the shear at the point in the length of the beam through which the ordinate is drawn.

We observe that the shear changes sign and is zero at the centre of the beam; we can conclude from this that the bending moment must have its maximum value at the centre, since shear = rate of change of bending moment, and if the shear is zero, the bending moment must have a turning value.

By sum-curving the shear curve from CD as base, the resulting curve is that of bending moment.

It is well carefully to consider the scales, for it is with these that difficulties often arise.

The scales given here apply to the original drawing, of which Fig. 31 is a reproduction somewhat under half full size.

For the length	1 in. = 2·5 ft.
For loads	1 in. = ·4 ton per ft.

Polar distance for the first sum curve, *i. e.*, the curve of shear—

$$= 4 \text{ ins.} = 4 \times 2{\cdot}5, \quad \text{or 10 horizontal units.}$$

Hence the scale of shear = $\cdot 4 \times 10$, or 4 tons to 1 in.

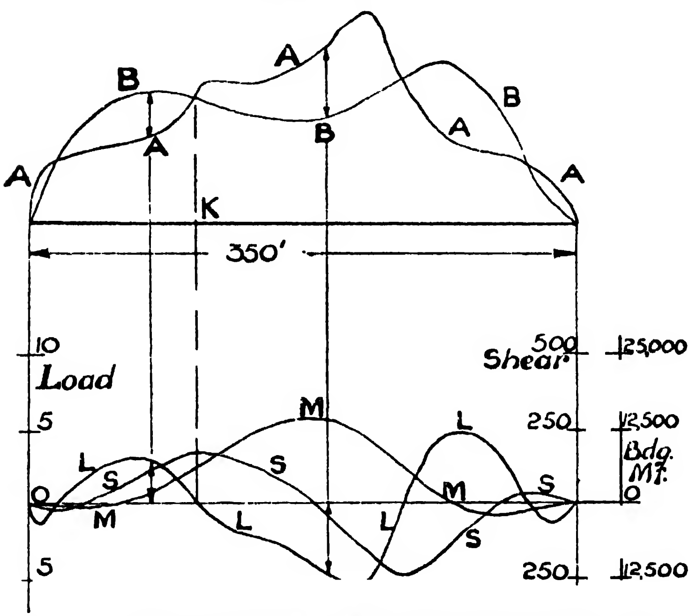


Fig. 32.—Shearing Force and Bending Moment on Ship's Hull.

Polar distance for the second sum curve = 4 ins. = 10 horizontal units.

Hence the scale of bending moment = $4 \times 10 = 40$
or 1 in. (vertically) = 40 tons. ft.

Reading according to these scales—

The maximum shear = 9·5 tons
and the maximum bending moment = 68 tons. ft.

Example 4.—In Fig. 32 AAA is the curve of weights or load distribution, and BBB the curve of buoyancy or upward water thrust

for a ship whose length is 350 ft., the scale of loads being indicated on the diagram.

Draw diagrams of shearing force and bending moment on the hull of the vessel and measure the maximum values of these quantities.

It is first necessary to construct the curve of loads to a straight line base, and to do this the differences between the curves AAA and BBB are set off from a horizontal, taken in our case below the original base line.

In this way the curve of loads LLL is obtained, the scale being shown to the left of the diagram.

By sum-curving this curve, the curve of shear SSS is obtained; the polar distance (not shown on the diagram) being taken as 50 horizontal units, so that the scale for the shear is 50 times the scale for the loads.

Sum-curving the curve SSS, the curve MMM, that of bending moment, is obtained (again the polar distance is 50 horizontal units).

Sections such as K, where the upward thrust of the water balances the downward force due to the weights, are spoken of as water-borne.

Reading our maximum values according to the proper scales, we find them to be—

$$\left.\begin{aligned}\text{Maximum shear} &= \underline{246 \text{ tons}}\\ \text{Maximum bending moment} &= \underline{14{,}300 \text{ tons ft.}}\end{aligned}\right\}$$

It should be noted that the last ordinate of both the shear curve and the curve of bending moment is zero; these results we should expect since the areas under the curves AAA and BBB must be equal, so that the shear at the end must be zero, and also the moments of these areas must be alike.

[In practice the maximum bending moment is found by such a formula as—

$$\text{Maximum bending moment} = \frac{\text{Weight} \times \text{length}}{\text{Constant}}$$

the constant for small boats being between 30 and 40, and for larger between 25 and 30.]

The Coradi Integraph.—A brief description of the Integraph, an instrument devised to draw mechanically the sum curve, can usefully be inserted at this stage.

It consists essentially of a carriage running on four milled wheels A (Fig. 33), a slotted arm C carrying the tracer B which is moved along the primitive curve, and the arm D which carries the pencil E which draws the sum curve.

As B is moved along the primitive curve, the slotted arm C slides about the pins G and P, thus altering its inclination to the horizontal. A parallel link motion ensures the movement of E parallel to the instantaneous position of C, the sharp-edged wheel F assisting in guiding the tracer bracket.

The principle of the instrument is not difficult to understand, and can be explained in a very few words.

The pole is at P, and the tops of the mid-ordinates are projected to the vertical through G by the horizontals like BG; parallels are then drawn to PG by the pencil E, the motion being continuous.

The polar distance can be varied as desired, by altering the position of the bracket carrying the pin P along the horizontal

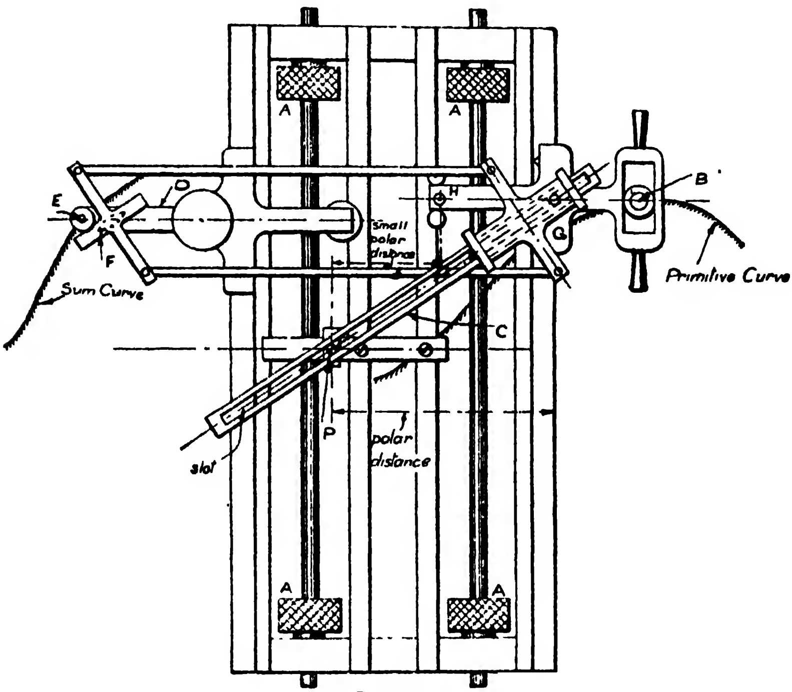


FIG. 33.

arm; and if an extremely small polar distance is found to be advisable, the pin H may be utilised instead of G.

Rules for Integration of Simpler Functions.—Since integration is the reverse of differentiation, many functions can be integrated by reversing the order of the steps in differentiation.

Integration of powers of x.—The first rule given in the work on differentiation of functions was—

$$\frac{d}{dx} x^n = nx^{n-1}.$$

To change into the integration form, we transpose $\frac{d}{dx}$: the "d" on the one side becomes $\int$ on the other side, to indicate the change *differencing* to *summing*, and the "dx" occurs on the top line of the other side of the equation.

Thus—
$$x^n = \int nx^{n-1}dx$$
or
$$\int x^{n-1}\,dx = \frac{1}{n}\,x^n + C$$

the reason for the presence of the constant term C being explained later.

It is a trifle simpler to write n in place of $n-1$, and therefore $n+1$ in place of n, so that—
$$\int x^n dx = \frac{1}{n+1}\;x^{n+1} + C.$$

Whereas, when *differentiating* a power of the I.V., *the power was reduced by* 1 in the process of differentiation; when *integrating, the power is increased by* 1.

E. g.,
$$\frac{d}{dx}\,x^5 = 5x^4$$
whereas
$$\int x^5 dx = \frac{1}{6}\,x^6 + C.$$

A special case occurs for which the above rule does not apply: for, let $n = -1$, then $\int x^{-1}dx$ should, according to the rule just given, be $\frac{1}{0}\,x^0$, but to this fraction no definite meaning can be assigned.

We know that
$$\frac{d}{dx}\log x = \frac{1}{x} = x^{-1}$$
$$\therefore \qquad \int x^{-1}\,dx = \log x + C$$

or, as it is sometimes written—
$$\int\frac{dx}{x} = \log x + C.$$

A constant multiplier before integration remains as such after;

thus
$$\int ax^n dx = \frac{a}{n+1}x^{n+1} + C.$$

Also an expression composed of terms can be integrated term by term, and the results added.

Thus, $\int (ax^n+b)\,dx$ can be written—

$$\int ax^n dx+\int b dx$$

i. e., $\int ax^n dx+\int bx^0 dx$ for $x^0 = 1$

its value being $\frac{a}{n+1}x^{n+1}+bx+C.$

Note.—Differentiation of a constant term gives zero, but the integration gives that constant multiplied by the I.V.

The reason for this will be apparent if we consider the statements from the graphical standpoint. The curve representing the equation $y = b$ is a horizontal straight line, and therefore the slope is zero $\left(i.\,e., \frac{db}{dx} = 0\right)$; but the area under the curve = the area of a rectangle = base × height = $x \times b$ (*i. e.*, $\int b dx = bx$).

Exponential Functions.—We have already proved that $\frac{de^x}{dx} = e^x$ (see p. 47); then by transposition of d and dx to the other side of the equation we obtain the statement $e^x = \frac{1}{d}e^x dx$, and since $\frac{1}{d}$ corresponds to $\int$ we may write this as $\int e^x dx = e^x+C$. Thus if we either differentiate e^x or integrate it we arrive at the same result; and e^x is the only function for which the differential coefficient and also the integral are the same as the function itself.

Carrying this work a step further, let us consider the integration of e^{bx}, and hence a^x :—

Now $\frac{d}{dx}ae^{bx} = abe^{bx}$ $\therefore \int ae^{bx}dx = \frac{a}{b}e^{bx}+C.$

To avoid confusion as to the placing of a and b we must reason in the following manner: The a is a constant multiplier of the whole function, and therefore remains so after integration; the b multiplies the I.V. only; and thus differentiation would cause it to multiply the result, whereas after integration it becomes a divisor. Great attention should be paid to the application of this rule, for unless care is exercised mistakes are very apt to creep in.

Example 5.—Find the value of $\frac{d}{dt}(15t^4-7t^{\cdot 9}+83)$ and also of $\int\left(60t^8-\frac{6\cdot 3}{t^{\cdot 1}}\right)dt.$

Differentiating the first expression—

$$\frac{d}{dt}(15t^4-7t^{\cdot 9}+83)$$
$$=60t^3-6\cdot 3t^{-\cdot 1}+0.$$

Integrating the second given expression—

$$\int\left(60t^3-\frac{6\cdot 3}{t^{\cdot 1}}\right)dt=\int 60t^3dt-\int 6\cdot 3t^{-\cdot 1}dt$$
$$=\frac{60\times t^4}{4}-\frac{6\cdot 3}{-\cdot 1+1}t^{-\cdot 1+1}+C$$
$$=15t^4-7t^{\cdot 9}+C.$$

Notice that although a function has been differentiated and the derivative integrated, the final expression is not exactly the same as the original, the constant term being represented only by C, where C may have any value. Further reference will be made to this point on p. 137.

Example 6.—If $pv^{1\cdot 32}=C$, find the value of $\int pdv$.

To express p in terms of v—

$$p=\frac{C}{v^{1\cdot 32}}=Cv^{-1\cdot 32}$$
$$\therefore\quad \int pdv=\int Cv^{-1\cdot 32}.dv=C\int v^{-1\cdot 32}dv$$
$$=C\times\frac{1}{-1\cdot 32+1}v^{-1\cdot 32+1}+K$$

(K being any constant)

$$=-\frac{C}{\cdot 32}v^{-\cdot 32}+K.$$

This result can be written in a slightly different form, if for C we write its value $pv^{1\cdot 32}$; then—

$$\int pdv=-\frac{pv^{1\cdot 32}\times v^{-\cdot 32}}{\cdot 32}+K$$
$$=-\frac{pv}{\cdot 32}=\underline{-3\cdot 125pv+K.}$$

Example 7.—Find $\int pdv$ when $pv=C$.

In this case— $p=Cv^{-1}$

$$\therefore\quad \int pdv=\int\frac{C}{v}dv=C\int\frac{dv}{v}=C\log v+K$$
$$=\underline{pv\log v+K.}$$

Example 8.—Find the value of $\int 17e^{2x}dx$.

$$\int 17e^{2x}dx=17\times\frac{1}{2}e^{2x}+C=\underline{8\cdot 5e^{2x}+C.}$$

Note that 17 is a constant multiplier throughout; 2 multiplies the I.V. and therefore appears as a divisor after integration; also the power of e remains exactly the same.

Example 9.—Find the value of $\int(40e^{5v}+v^{5\cdot4})dv$.

$$\int(40e^{5v}+v^{5\cdot4})dv = \int 40e^{5v}dv+\int v^{5\cdot4}dv \text{ (separating the terms)}$$
$$= \left(40\times\frac{1}{5}e^{5v}\right)+\frac{1}{6\cdot4}v^{6\cdot4}+C$$
$$= \underline{8e^{5v}+\cdot156v^{6\cdot4}+C.}$$

Example 10.—Find an expression for $\int a^x dx$, and apply the result to determine the value of $\int 12\times(5)^{4x}dx$.

From our previous work we know that—

$$\frac{d}{dx}a^x = a^x.\log a.$$

$$\therefore \qquad \int a^x dx = \underline{\frac{1}{\log a}.a^x+C.}$$

Hence
$$\int 12\times5^{4x}dx = \left(12\times\frac{1}{4}\times\frac{1}{\log 5}\times5^{4x}\right)+C$$
$$= \frac{3}{1\cdot609}\times5^{4x}+C \qquad \log_e 5 = 1\cdot609$$
$$= \underline{1\cdot864\times5^{4x}+C.}$$

Note.—It would be quite incorrect to multiply 1·864 by 5 and express the result as $9\cdot32^{4x}$.

Alternatively, the result might have been arrived at in the following manner—

$$\int 12\times5^{4x}dx = 12\int(5^4)^x dx$$
$$= 12\times\frac{1}{\log 625}\times(5^4)^x+C$$
$$= \frac{12}{6\cdot44}\times5^{4x}+C \qquad (\log 625 = 6\cdot44)$$
$$= \underline{1\cdot864\times5^{4x}+C.}$$

Exercises 12.— On Graphic Integration.

1. The acceleration of a slider at various times is given in the table. By graphic integration obtain the velocity and displacement curves to a time base, indicating clearly your scales.

Time . .	0	·008	·016	·02	·028	·036	·044	·048	·06
Acceleration	0	75	87·5	87·5	87·5	87·5	87·5	83	0

·068	·072	·084	·10	·108	·12
78	85	87·5	87·5	83	0

2. An acceleration diagram on a time base has an area of 4·7 sq. ins. The base of the diagram is 2·5 ins. and represents 25 secs. The acceleration scale is 1 in. = 3 ft. per sec.². If the velocity at the beginning is 11 ft. per sec., find the velocity in ft. per sec. at the end of the 25 secs.

3. A rectangular barge is loaded symmetrically in still water. The curve of loading is a triangle with apex at the centre, and the curve of buoyancy is a rectangle. Draw diagrams of shearing force and bending moment on the barge.

4. The curves of loads for a ship 350 ft. long is as given in the table. Plot this, and by graphic integration obtain the curves of shearing force and bending moment.

Distance from one end (ft.)	0	7	10	35	56	84	102
load (tons per foot) . . .	0	·3	0	−2·6	−3·1	−2·3	0

112	133	161	196	210	237	260	280	315	330	350
1·6	2·3	3·15	5·2	5·7	0	−4·5	−4·95	0	·9	0

5. The table gives the values of the pressure and volume for the complete theoretical diagram for a triple expansion engine.

v	0	1	2	4	6	8	10	12
p	240	240	120	60	40	30	24	20

Find the initial pressure in each cylinder in order that the work done per cycle may be the same for each.

(*Hint.*—Divide the last ordinate of the sum curve into three equal parts, draw horizontals through these points of section to meet the sum curve, and from these points of contact erect perpendiculars to cut the expansion line.)

6. A body weighing 3000 lbs. was lifted vertically by a rope, there being a damped spring balance to indicate the pulling force F lbs. of the rope. When the body had been lifted x ft. from its position of rest, the pulling force was automatically registered as follows :—

x	0	20	40	65	75	95	110	140
F	8000	7950	7850	7500	7400	6800	6400	4000

Find the work done on the body when it has risen 80 ft. How much of this is potential energy and how much is kinetic energy?

Find also the work done when it has risen 140 ft.

7. The current from a battery was measured at various times, with the following results :—

Time (hours) . .	0	1	2	3	5	6	7·8	9	10	12	14	15
Current (amperes)	25	28	32·8	37	39·6	39·5	36	32	29	24	25·3	27

If its capacity is measured by $\int C dt$, find the capacity in ampere hours.

8. The following are the approximate speeds of a locomotive on a run over a not very level road. Draw a curve showing the distance run up to any time.

Time (mins. and secs.)	0	1	2.15	6.15	9.22	11.45	14.26	16.35	20.52
Speed (miles per hr.)	0	6	10	18·2	22·8	25·5	28	29·2	28·6

9. The load curve at a large central station can be constructed from the following data :—

Time (hours) . . .	0	1	2	3	4	5	6	7	7·5	8	9	10	11	12	1	2	3	4	5	5·5	6	7	8	9	10	11	12
Load (1000 amperes)	3·5	1	1	2	1	6·4	14	17	17·8	16·4	11·3	8·7	8·2	7·8	8	7·6	8·7	12·5	19	23·3	21	12·4	11	10·5	9·6	9	6

Find the total number of ampere hours supplied in the 24 hours.

10. The velocity of a three-phase electric train, with rheostatic control, at various times, was found as in the table :—

Time (secs.) .	0	26·6	66·6	80·1	99
Velocity (ft. per sec.) .	0	40	40	37·3	0

Draw the space-time curve and find the total distance covered in the 99 seconds.

On the Integration of the Powers of x and of the Exponential Functions.

11. What is the significance of the symbols $\int$ and dx in the expression $\int x^2 dx$?

Integrate, with respect to x, the functions in Examples 12 to 27

12. $4x^{1.63}$. **13.** $70·15$. **14.** $\frac{3}{x^3}$ **15.** $\frac{x^8}{x^{n+1}}$ **16.** $e^{.7x}$.

17. $x^9 - \frac{10}{x} + 14$. **18.** $\frac{3·47}{x^{-.04}} - 5x$. **19.** $4x^e + \frac{1}{4}e^x$.

20. $12e^{9x-8}$. **21.** $\dfrac{15.7}{e^{1.2-2.4x}}$. **22.** $.17e^{1.4}$. **23.** $\dfrac{b(x^6)^{.4}}{cx^{1.89}}$.

24. $2.54x^{-.16}-8.2x^{-1}+\dfrac{7.04}{e^{2.6x}}+1.13$. **25.** $\dfrac{82e^{2x}\times e^{-5x}}{(7e^{2.4x})^3}$.

26. $(e^3)^{.17x}-x^{-.23}+\dfrac{5.03x^{2.04}}{e^{.9}}$. **27.** $.94x^{.18}\cos\theta-\dfrac{1.76x^6}{e^{16-.8x}}$.

Find the values of the following—

28. $\int v^5dv$. **29.** $\int\dfrac{du}{u^4}$. **30.** $\int 35dt$. **31.** $\int e^{3s-4}ds$.

32. $\int pdv$ when $pv^{1.17}=C$. **33.** $\int 14\times 2^s ds$.

34. $\int(4x^{-3}+5x+17x^3-8)dx$. **35.** $\int 3.1^t dt$. **36.** $17\int\dfrac{dp}{e^{3p}}$.

37. $\int(e^{4t}+e^{-bt}-e)dt$. **38.** $\iint 32.2(dt)^2$. **39.** $2.1\iint x^{-5}(dx)^2$.

40. Solve the equation $\dfrac{dp}{dv}=-n\dfrac{p}{v}$.

41. In connection with the flow of air through a nozzle, if x is the distance outwards from the nozzle and v is the velocity there, $v\propto\dfrac{1}{x}$. Also δA (an element of area of flow) $=K\delta x\sqrt{x}$. The added momentum for the small element considered $=\delta M=v^2\delta A$. Show that M, the total increment to the momentum, can be written $C-\dfrac{D}{\sqrt{x}}$ where C and D are constants.

Trigonometric Functions.—We have previously seen that the derived curve of either the sine curve or the cosine curve is the primitive curve itself transferred *back* a horizontal distance of one-quarter of the period. Conversely, then, we may state that the sum curve of either the sine or the cosine curve is the curve itself moved *forward* for a distance corresponding to one-quarter of the period. In other words, integration does not alter the form of the curve. Taking the case of the sine curve as the primitive, we see, on reference to Fig. 34, that if this curve is shifted forward for one-quarter period the resulting curve is the cosine curve inverted; or expressing in algebraic language, whilst the equation of the primitive curve is $y=\sin x$, that of the sum or integral curve is $y=-\cos x$. Thus, $\int\sin x dx=-\cos x$. In like manner it could be shown that $\int\cos x dx=\sin x$. For emphasis, the differentiation and the integration of sine x and cosine x are repeated here—

$$\frac{d}{dx}\sin x=\cos x \qquad \int\sin x\,dx=-\cos x+C$$

$$\frac{d}{dx}\cos x=-\sin x \qquad \int\cos x\,dx=\sin x+C.$$

Note.—When *differentiating the cosine* the *minus sign* appears in the result; when *integrating the sine* the *minus sign* appears; it is important to get a good grip of these statements, and the consideration of them from the graphic aspect is a great help in this respect.

To extend the foregoing rules—

$$\int \cos x dx = \sin x + C$$

$$\therefore \quad \int \cos (ax+b)dx = \frac{1}{a} \sin (ax+b) + C$$

$$\int \sin x dx = -\cos x + C$$

$$\therefore \quad \int \sin (ax+b)dx = -\frac{1}{a} \cos (ax+b) + C.$$

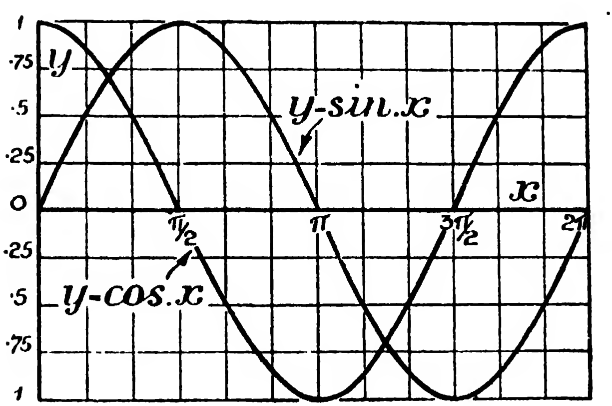


FIG. 34.

Thus the angle remains the same after integration just as it would after differentiation, but the constant multiplier a of the I.V. becomes a divisor.

To integrate $\sec^2 x$ with regard to x we call to mind the differentiation of $\tan x$, viz., $\frac{d}{dx} \tan x = \sec^2 x$. Accordingly $\int \sec^2 x dx = \tan x + C$.

Extending this to apply to the more general case—

$$\int \sec^2 (ax+b)dx = \frac{1}{a} \tan (ax+b) + C$$

In like manner—

$$\int \operatorname{cosec}^2 (ax+b)dx = -\frac{1}{a} \cot (ax+b) + C.$$

Other two standard integrals are added here, the derivation of which will be considered in the next chapter.

$$\int \tan x\,dx = -\log\cos x + C$$
$$\int \cot x\,dx = \log\sin x + C.$$

To verify these we might work from the right-hand side and differentiate. Dealing with the former—

$$\frac{d}{dx}(-\log\cos x) = -\frac{d}{dx}\log u \qquad \text{if } u = \cos x \text{ whence}$$
$$= -\frac{d\log u}{du}\times\frac{du}{dx} \qquad \frac{du}{dx} = -\sin x$$
$$= -\frac{1}{u}\times -\sin x$$
$$= \frac{\sin x}{\cos x} = \tan x$$
$$\therefore \quad \int \tan x\,dx = -\log\cos x + C.$$

Example 11.—Find the value of $\int(5-\sin 4t)dt$.

$$\int(5-\sin 4t)dt = \int 5dt - \int \sin 4t\,dt$$
$$= 5t - \left(\frac{1}{4}\times -\cos 4t\right) + C$$
$$= \underline{5t + \frac{1}{4}\cos 4t + C.}$$

Example 12.—Evaluate $\int \sin(5-4t)dt$.

$$\int \sin(5-4t)dt = -\frac{1}{4}\times -\cos(5-4t) + C = \underline{\frac{1}{4}\cos(5-4t) + C.}$$

Example 13.—If a force P is given by $P = 36{\cdot}4\sin(100s - {\cdot}62)$, find the value of $\int P\,ds$.

$$\int P\,ds = \int 36{\cdot}4\sin(100s - {\cdot}62)ds = \frac{36{\cdot}4}{100}\times -\cos(100s - {\cdot}62) + C$$
$$= \underline{-{\cdot}364\cos(100s - {\cdot}62) + C.}$$

Example 14.—Find the value of

$$\int\left[7{\cdot}2s^4 - \frac{3}{s} + \frac{15s^{1\cdot7}}{s^{\cdot9}} + 12\cos(4-3s)\right]ds.$$

The expression $E = 7{\cdot}2s^4 - 3s^{-1} + 15s^{\cdot8} + 12\cos(4-3s)$

$$\therefore \quad \int E\,ds = \left(7{\cdot}2\times\frac{1}{5}s^5\right) - 3\log s + \frac{15}{1{\cdot}8}s^{1\cdot8} + \left(12\times -\frac{1}{3}\sin(4-3s)\right) + C$$
$$= \underline{1{\cdot}44s^5 - 3\log s + 8{\cdot}33s^{1\cdot8} - 4\sin(4-3s) + C.}$$

Example 15.—If $R = 11\sec^2(3-4{\cdot}7v)$, find $\int R\,dv$.

$$\int R\,dv = \int 11\sec^2(3-4{\cdot}7v)\,dv = \frac{11}{-4{\cdot}7}\tan(3-4{\cdot}7v)+C$$
$$= \underline{-2{\cdot}343\tan(3-4{\cdot}7v)+C.}$$

Exercises 13.—On Integration of Trigonometric Functions.

Integrate, with respect to x, the functions in Nos. 1 to 10.

1. $3\sin 4x$. 2. $-5{\cdot}18\cos(3-3x)$. 3. $7\sec^2\left(3-\frac{1}{7}x\right)$

4. $x^{-\cdot 012}-\cdot 14\cos(\cdot 05-\cdot 117x)$. 5. $e^{5.4x}+5\sin(b+ax)$.

6. $9{\cdot}45\sin 8t$. 7. $-3{\cdot}08\sin 2(2{\cdot}16x-4{\cdot}5)$.

8. $9e^{\cdot 7x}+\frac{3{\cdot}47}{x^5}-1{\cdot}83\tan x$.

9. $4{\cdot}27\sin\left(\frac{3x-2{\cdot}8}{7}\right)+\cdot 2\cos 9x-4x^{1.74}+3^{2x+5}$.

10. $2\sin^2 x-2{\cdot}91\sin\left(\frac{\pi}{4}-3{\cdot}7x\right)+2\cos^2 x-14{\cdot}2\operatorname{cosec}^2\frac{3\pi x}{5}$.

11. The acceleration of a moving body is given by the equation—

$$a = -49\sin(7t-\cdot 26).$$

Find expressions for the velocity and the space, the latter being in terms of the acceleration.

12. If $\frac{d^2x}{d\theta^2} = 4\pi^2n^2r\left(\cos\theta+\frac{\cos 2\theta}{m}\right)$, find the values of $\frac{dx}{d\theta}$ and x. (x is a displacement of the piston in a steam-engine mechanism.)

13. Find the value of $\int[5^{2p}+\cos(3{\cdot}7-7{\cdot}2p)]dp$.

14. If $v = 117\sin 6t-29{\cdot}4\cos 6t$, find the value of $\int v\,dt$.

Indefinite and Definite Integrals.—The integrals already given, although correct, are not complete. If an integral is to denote an area some boundaries must be known; and nothing was said about the limits to be ascribed to x (or s, as the case might be) in the foregoing, so that we were in reality dealing with indefinite areas or integrals. To indicate that a portion of the area may be dispensed with in certain cases (when the boundaries are stated) a constant C is introduced on the R.H.S. of the equation, *i. e.*, $\int x^3dx$ would be written $\frac{x^4}{4}+C$.

As soon as the integral, and therefore the area, is made definite it will be observed that C vanishes.

If $\int x^3dx$ is to equal $\frac{1}{4}x^4+C$, $\frac{d}{dx}\left(\frac{1}{4}x^4+C\right)$ should equal x^3; and this is the case for—

$$\frac{d}{dx}\left(\frac{1}{4}x^4+C\right) = \frac{d}{dx}\frac{1}{4}x^4+\frac{d}{dx}.C = x^3+0 = x^3$$

(C being independent of x).

Cf. Example 5, p. 129; the constant in that case being 83.

It is therefore advisable to add the constant in all examples on integration; in many practical examples the determination of the value of the constant is an important feature, and therefore its omission would invalidate the results obtained.

In the list of a few of the simpler standard integrals collected together here for purposes of reference and by way of revision the constant is denoted by C.

$$\int(ax^n+b)dx = \frac{a}{n+1}x^{n+1}+bx+C$$

$$\int ax^n dx = \frac{a}{n+1}x^{n+1}+C$$

$$\int b dx = bx+C$$

$$\int\frac{dx}{x} = \log x+C$$

$$\int\frac{dx}{ax+b} = \frac{1}{a}\log(ax+b)+C$$

$$\int ae^{bx}dx = \frac{a}{b}e^{bx}+C$$

$$\int e^x dx = e^x+C$$

$$\int a^x dx = \frac{1}{\log a}\times a^x+C$$

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+C$$

$$\int \sin x dx = -\cos x+C$$

$$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+C$$

$$\int \cos x dx = \sin x+C$$

$$\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+C$$

$$\int \sec^2 x dx = \tan x+C$$

$$\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)+C$$

$$\int \operatorname{cosec}^2 x dx = -\cot x + C$$

$$\int \tan(ax+b)dx = -\frac{1}{a}\log\cos(ax+b)+C$$

$$\int \tan x dx = -\log(\cos x)+C$$

$$\int \cot(ax+b)dx = \frac{1}{a}\log\sin(ax+b)+C$$

$$\int \cot x dx = \log(\sin x)+C.$$

Method of Determining the Values of Definite Integrals. —It is stated on p. 118 that the limiting value of $\Sigma y\delta x$ is $\int ydx$, and that the area of ABCD, Fig. 27, is given by the value of $\int_a^b ydx$. We can now show how such a definite integral may be evaluated.

The area of the strip EFGH may be written as an element of the area under the curve, say $\delta\bar{A}$, so that $\delta\bar{A} = y\delta x$ and $\frac{\delta\bar{A}}{\delta x} = y$, approximately, or $\frac{d\bar{A}}{dx} = y$, exactly. Thus $\bar{A}$ is a function which, when differentiated with respect to x, gives y, or, in other words, $\bar{A} = \int ydx$. Now suppose that y is a function of x such that $\frac{d\phi(x)}{dx} = y$; then $\phi(x)+c = \int ydx$ and $\bar{A} = \phi(x)+C$.

In the case of the area considered, the measurement of area is to commence when $x = a$, *i.e.* it is to have the value 0 when $x = a$

Hence $0 = \phi(a)+c$, $C = -\phi(a)$ and $\bar{A} = \phi(x)-\phi(a)$.

Also, denoting the area ABCD by A, we observe that $\bar{A}$ is to have the value A when $x = b$, and thus $A = \phi(b)-\phi(a)$.

E.g., if $y = x^2$, $a = 2$ and $b = 4$, $\phi(x) = \frac{x^3}{3}$

and $$\int_2^4 x^2dx = \left(\frac{4^3}{3}\right)-\left(\frac{2^3}{3}\right) = 18\tfrac{2}{3}$$

meaning that if the curve $y = x^2$ were plotted, and the area between the curve, the x axis and the ordinates through $x = 2$ and $x = 4$ found, its value would be $18\frac{2}{3}$ sq. units.

It will be noticed that C vanishes, and hence when dealing with definite integrals it is usual to omit it altogether.

For brevity, $\int_2^4 x^2dx$ is written $\left(\frac{x^3}{3}\right)_2^4$, which on expansion reads—

$$\left(\frac{4^3}{3}-\frac{2^3}{3}\right), \ i.e., \ \frac{56}{3}.$$

Example 16.—Find the value of the definite integral $\int_{\cdot 1}^{\cdot 4} 4e^{3x}dx$.

$$\int_{\cdot 1}^{\cdot 4} 4e^{3x}dx = \left(\frac{4}{3}e^{3x}\right)_{\cdot 1}^{\cdot 4} = \frac{4}{3}\left(e^{3x}\right)_{\cdot 1}^{\cdot 4}$$

$$= \frac{4}{3}(e^{3\times\cdot 4}-e^{3\times\cdot 1})$$

$$= \frac{4}{3}(e^{1\cdot 2}-e^{\cdot 3})$$

$$= \frac{4}{3}(3\cdot 3201-1\cdot 3499)$$

$$= \underline{2\cdot 625.}$$

Example 17.—Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}}(5\cos 4x+7)\,dx$$

$$\int_0^{\frac{\pi}{2}}(5\cos 4x+7)\,dx = \left(\frac{5}{4}\sin 4x+7x\right)_0^{\frac{\pi}{2}}$$

$$= \left(\frac{5}{4}\sin\frac{4\pi}{2}+\frac{7\pi}{2}-\frac{5}{4}\sin 4\times 0-7\times 0\right)$$

$$= \left(0+\frac{7\pi}{2}-0-0\right)$$

$$= \underline{\frac{7\pi}{2} \text{ or } 11.}$$

Example 18.—Find the value of

$$\frac{\int_2^4 5x^5dx}{\int_2^4 5x^3dx}.$$

The expression

$$= \frac{5\int_2^4 x^5dx}{5\int_2^4 x^3dx} = \frac{\left(\frac{x^6}{6}\right)_2^4}{\left(\frac{x^4}{4}\right)_2^4}$$

$$= \frac{\frac{1}{6}(4^6-2^6)}{\frac{1}{4}(4^4-2^4)} = \underline{11\cdot 2.}$$

Notice that no cancelling takes place, beyond that concerning the constant multiplier 5, until the values (4 and 2) have been substituted in place of x. In other words, it would be quite wrong to say—

$$\frac{\left(\frac{x^6}{6}\right)_2^4}{\left(\frac{x^4}{4}\right)_2^4} = \left(\frac{2}{3}x^2\right)_2^4 = \frac{2}{3}(12) = 8$$

Example 19.—The total range of an aeroplane in miles can be obtained from the expression $-\int_1^q \frac{m}{q}dq$ where m = pound-miles per lb. of petrol, and $q = \frac{\text{loading at any time}}{\text{initial loading}}$.

Taking $q = \cdot 6$ and $m = 4000$, find the total range.

$$-\int_1^q \frac{m}{q}\,dq = -m\int_1^q \frac{dq}{q} = -m\left(\log q\right)_1^q$$
$$= -m(\log q - \log 1)$$
$$= -m \log q.$$

Now if $q = \cdot 6$, $\log q = \bar{1}\cdot 4892 = -\cdot 5108$.

Hence the range $= 4000 \times \cdot 5108 = \underline{2043 \text{ miles}}$.

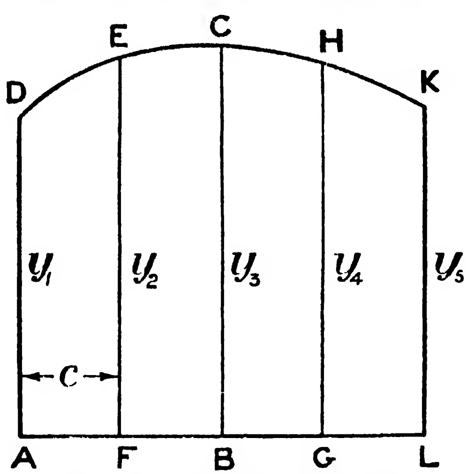


FIG. 35.—Proof of Simpson's Rule.

Proof of Simpson's Rule for the Determination of Areas of Irregular Curved Figures.—This rule, given on p. 310 of Part I, states that—

$$\text{Area} = \frac{\text{length of one division of the base}}{3}\left\{\begin{array}{l}\text{first + last ordinate +}\\ 4\Sigma \text{ even ordinates +}\\ 2\Sigma \text{ odd ordinates.}\end{array}\right\}$$

It is now possible to give the proof of this rule.

Let us deal with a portion of the full area to be measured, such as ABCD in Fig. 35. Let the base AB $= 2c$.

Let the equation of the curve DEC be $y = A+Bx+Cx^2$, so that DEC is a portion of some parabola.

We can assume that the origin is at F, and therefore the abscissæ of D, E and C are $-c$, o and $+c$ respectively.

$$\begin{aligned}
\text{Hence}\quad AD &= y_1 = A+B(-c)+C(-c)^2 &&= A-Bc+Cc^2 \\
FE &= y_2 = A+B(0)+C(0)^2 &&= A \\
BC &= y_3 = A+B(c)+C(c)^2 &&= A+Bc+Cc^2.
\end{aligned}$$

$$\begin{aligned}
\text{Now the area ABCD} &= \int_{-c}^{+c} y\,dx \\
&= \int_{-c}^{+c} (A+Bx+Cx^2)dx \\
&= \left[Ax+\frac{Bx^2}{2}+\frac{Cx^3}{3}\right]_{-c}^{+c} \\
&= Ac+\frac{Bc^2}{2}+\frac{Cc^3}{3}+Ac-\frac{Bc^2}{2}+\frac{Cc^3}{3} \\
&= 2Ac+\frac{2Cc^3}{3} \\
&= \frac{c}{3}\{6A+2Cc^2\} \\
&= \frac{c}{3}\{A-Bc+Cc^2+4A+A+Bc+Cc^2\} \\
&= \frac{c}{3}\{y_1+4y_2+y_3\}
\end{aligned}$$

Imagine now another strip of total width $2c$ added to the right of BC; the double width being chosen, since there must be an even number of divisions of the base.

Then if GH $= y_4$ and LK $= y_5$

$$\begin{aligned}
\text{Area of BLKC} &= \frac{c}{3}\{y_3+4y_4+y_5\} \\
\text{or area of ALKD} &= \frac{c}{3}\{y_1+4y_2+y_3+y_3+4y_4+y_5\} \\
&= \frac{c}{3}\{y_1+y_5+4(y_2+y_4)+2(y_3)\}.
\end{aligned}$$

If a strip of width $= 2c$ is added to the right—

$$\text{Area} = \frac{c}{3}\{y_1+y_7+4(y_2+y_4+y_6)+2(y_3+y_5)\}.$$

Or, in general—

$$\text{Area} = \frac{c}{3}\{\text{first}+\text{last}+4\Sigma\text{ even}+2\Sigma\text{ odd}\}.$$

Exercises 14.—On the Evaluation of Definite Integrals.

Find the values of the definite integrals in Nos. 1 to 7.

1. $\int_{1\cdot 02}^{1\cdot 16} x^3 dx.$ 2. $\int_{1\cdot 7}^{2\cdot 4} \frac{dv}{v}$ 3. $\int_{\cdot 2}^{\cdot 88} e^{4s} ds.$

4. $\int_0^{\frac{\pi}{3}} 5\cdot 1 \sin \cdot 2\theta d\theta.$ 5. $\int_{\cdot 2}^{\cdot 48} 2t^{1\cdot 7} dt.$

6. $\int_{1\cdot 2}^{2\cdot 7} 5\cdot 2^{4\cdot 1x} dx.$ 7. $\dfrac{\int_1^2 x^{1\cdot 74} dx}{\int_1^2 x^{\cdot 4}}.$

8. The change in entropy of a gas as the absolute temperature changes from 643 to 775 is given by $\int_{643}^{775} \cdot 85 \frac{d\tau}{\tau}$. Find this change.

9. If $H = \frac{2l}{p}\int_0^{\frac{\pi}{2}} \sin \theta d\theta$, find the value of H.

10. The average useful flux density (for a 3-phase motor)

$$= B = \frac{1}{\pi}\int_{\frac{\pi}{12}}^{\frac{11}{12}\pi} B_{max} \sin \theta d\theta. \quad \text{Find B in terms of } B_{max}.$$

11. Express $\sin at \cos bt$ as the sum of two terms and integrate with regard to t. If a is $\frac{2\pi}{T}$ and b is $3a$, what is the value of the integral between the limits 0 and T?

12. If $h = \frac{v_1 r_1^2}{g}\int_{R_1}^{R_2} \frac{dr}{r^3}$, find h.

13. Given that $EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2} - Px$. Also that $\frac{dy}{dx} = 0$ when $x = l$, and $y = 0$ when $x = 0$ and also when $x = l$; find the value of P and an expression for y.

14. If $M = \frac{wx^2}{2}$, $\frac{M}{I} = E\frac{d^2y}{dx^2}$, $\frac{dy}{dx} = 0$ and also $y = 0$ when $x = l$, find an expression for y.

15. Given that $M = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right) - K$, $\frac{M}{IE} = \frac{d^2y}{dx^2}$. Also $\frac{dy}{dx} = 0$ when $x = \frac{l}{2}$, and $y = 0$ when $x = \pm\frac{l}{2}$; find an expression for y. (The case of a fixed beam uniformly loaded.)

16. Find the value of $\int_0^l (lx - x^2)^2 dx.$

17. Evaluate $3\int_{\cdot 8l}^{l} x(l^2-2lx+x^2)dx$, an integral occurring in a beam problem.

18. If $Q = \int_0^h q\,dx$ and $q = \frac{1-\sin\phi}{1+\sin\phi}wx$, find Q, the total horizontal thrust on a retaining wall of height h, w being the weight of 1 cu. ft. of earth, and ϕ the angle of repose of the earth.

19. Find the area between the positive portion of the curve $y = 3x-4x^2+11$ and the axis of x, and compare with the area of the surrounding rectangle.

20. Evaluate $\int_{4\cdot 6}^{16\cdot 8} p\,dv$ when $pv^{1\cdot 37} = 594$.

21. If $\frac{d^2y}{dx^2} = 6x^{1\cdot 4}-\frac{5}{x^2}$: $\frac{dy}{dx} = 10\cdot 5$ when $x = 1$, and $y = 14$ when $x = 2$, find an expression for y in terms of x.

22. Evaluate $\int_{1\cdot 47}^{2\cdot 31}\frac{4}{13-5x}dx$.

23. Find the value of n, given by the relation $n = \int_0^l \frac{Ne^{ax}dx}{l}$.

24. The total centrifugal force on a ring $= \int_{R_2}^{R_1}\frac{2\pi w V_1^2 r^2 dr}{R_1^2}$; find an expression for the force.

25. The area of a bending moment diagram in a certain case was given by $-\int_0^l\left(\frac{1}{2}al-\frac{a^2}{2l}\right)da$; find the value of this area.

26. H, the horizontal thrust on a parabolic arch,

$$= \frac{5wl^2}{8r}\int_0^{\frac{1}{2}}(x^2-2x^3+x^4)dx.$$

Find an expression for H.

27. The work done by an engine working on the Rankine cycle with steam kept saturated $= \int_{\tau_2}^{\tau_1}\frac{L}{\tau}d\tau$.

Find the work done if the temperature limits are 620° F. and 800° F. (both absolute), and $L = 1437-\cdot 7\tau$.

28. Evaluate $\int_1^{2\cdot 4}[e^{\cdot 8s}-\sin(2s-7)]ds$.

29. Find the value of n, the frequency of transverse vibrations of a beam simply supported at its ends and uniformly loaded with w tons per foot run, when the equation of the deflected form is—

$$y = \frac{w}{24EI}(x^4-2lx^3+l^3x)$$

and

$$n^2 = \frac{g\int_0^l y\,dx}{4\pi^2\int_0^l y^2dx}$$

30. From Dieterici's experiments we have the following relations—

If s = specific volume of liquid ammonia

and c = specific heat of liquid ammonia

then for temperatures above 32° F.—

$$c = 1 \cdot 118 + \cdot 001156(t - 32)$$

and $$s = \int_0^t c\,dt.$$

Find s when t = 45° F.

31. If $Q = \frac{\pi d l \rho}{8 l \mu}\int_0^s (s^2 - x^2)dx$, Q being the leakage of fluid past a well-fitting plug, find its value.

32. The total ampere conductors per pole due to the three windings in a railway motor $= \frac{3}{2}C\sqrt{2}\int_0^l \frac{2h}{\pi} A_1 \sin\frac{\pi x}{l}dx.$

Evaluate this integral.

33. For a viscous fluid flowing through a narrow cylindrical tube of radius r, the quantity Q is given by the formula—

$$Q = \int_0^a \frac{\pi f(a^2 - r^2) r\,dr}{2\mu}$$

where μ is the coefficient of viscosity.

Find the value of Q.

34. If $u = \frac{4\pi l}{10}\left[\int_x^{x_1}\frac{dx}{\pi x} + \frac{y}{2x}\right]$ find its value.

35. Find the area between the curve $y = 2x^3 - 11x^2 - 21x + 90$, the axis of x, and the ordinates through $x = -3$ and $x = 7$.

36. Evaluate $Q = \frac{8B}{H}\int_0^H (H - h)h^{\frac{1}{2}}dh$ when $B = 1 \cdot 5$ and $H = \cdot 6$.

37. Find the value of $\int_{-2}^{2 \cdot 5}(5a - 11)^2 da$.

38. Find the time t taken to discharge over a weir of length b, the level of the water falling from 1·2 ft. to ·5 ft., from

$$t = \int_{\cdot 5}^{1 \cdot 2}\frac{A\,dh}{Kbh^{\frac{3}{2}}}$$

when $A = 20$, $K = 3$, and $b = 5 \cdot 5$.

39. Evaluate $\int_{\frac{1}{600}}^{\frac{1}{300}} 40 \sin\left(100\pi t + \frac{\pi}{6}\right)dt.$

CHAPTER VI

FURTHER METHODS OF INTEGRATION

By the use of the rules enumerated in the previous chapter it is possible to perform any integration by a graphic method and the integration of the simpler functions by algebraic processes. Whilst the graphic integration is of universal application, it at times involves much preliminary arithmetical work, which it is tedious to perform, so that it is very frequently the better plan to resort to a somewhat more difficult, though shorter, algebraic method. For the more complex functions, then, a choice has to be made between the two methods of attack; the fact being borne in mind that only in cases where *definite integrals* are concerned does the graphic method of integration compare favourably with the algebraic.

It is therefore advisable to introduce new processes and artifices to be employed for the algebraic integration of difficult functions; and whilst it is not absolutely essential that all these forms should be remembered, it is well that the various types should be considered, so that they may be recognised when they occur.

It is impossible to deal here with every kind of integral likely to be encountered; all that can be done is to develop the standard forms which cover a wide range, and to leave them to suggest forms for particular cases.

Integration by the Aid of Partial Fractions.—Many complex fractions can be split up into simpler or partial fractions, to which the simple rules of integration may be applied. Thus if we are asked to integrate, with respect to x, the fraction $\frac{8x-30}{2x^2-15x+28}$, we soon discover that we are unable to perform this operation with only the knowledge of integration acquired from the previous chapter.

If, however, we break the fraction up, in the manner explained in Part I, Chap. XII, we find that the integration resolves itself into that of two simple fractions.

Thus— $\dfrac{8x-30}{2x^2-15x+28} = \dfrac{2}{x-4}+\dfrac{4}{2x-7}$ (see Part I, p. 453).

Hence— $\displaystyle\int\frac{8x-30}{2x^2-15x+28}dx = \int\frac{2}{x-4}dx+\int\frac{4}{2x-7}dx$

$$= 2\log(x-4)+\frac{4}{2}\log(2x-7)+\log C$$
$$= \log(x-4)^2+\log(2x-7)^2+\log C$$
$$= \log\{C(x-4)^2(2x-7)^2\}.$$

[Note that log C may be written to represent the constant in place of C alone; and it can then be combined with the other logs.]

Example 1.—Find the value of $\displaystyle\int\frac{dx}{x^2-a^2}$

Let
$$\frac{1}{x^2-a^2} = \frac{A}{(x-a)}+\frac{B}{(x+a)}$$
$$= \frac{A(x+a)+B(x-a)}{x^2-a^2}.$$

Equating numerators—
$$1 = A(x+a)+B(x-a).$$

Let $x = a$, then $1 = A(2a)+0$

and $A = \dfrac{1}{2a}$.

Let $x = -a$, then $1 = 0+B(-2a)$

and $B = -\dfrac{1}{2a}$

$$\therefore \quad \frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right).$$

Hence— $\displaystyle\int\frac{dx}{x^2-a^2} = \frac{1}{2a}\left\{\int\frac{dx}{x-a}-\int\frac{dx}{x+a}\right\}$

$$= \frac{1}{2a}\{\log(x-a)-\log(x+a)+\log C\}$$
$$= \underline{\frac{1}{2a}\log\frac{C(x-a)}{(x+a)}}$$

This is a standard form.

A rather more general result may be deduced from it.

Example 2.—To find $\displaystyle\int\frac{dx}{(x+a)^2-b^2}$.

Let $(x+a) = X$

Then— $\displaystyle\int\frac{dx}{(x+a)^2-b^2} = \int\frac{dX}{X^2-b^2}$

$$= \frac{1}{2b}\log\frac{C(X-b)}{(X+b)}$$
$$= \underline{\frac{1}{2b}\log\frac{C(x+a-b)}{(x+a+b)}}.$$

Explanation.

$$x+a = X$$
$$\therefore \quad \frac{d}{dx}(x+a) = \frac{dX}{dx}$$
$$1 = \frac{dX}{dx}$$

and thus for dx we may write dX.

Integration by the Resolution of a Product into a Sum.—A product cannot be integrated directly; but when the functions are trigonometric the product can be broken up into a sum or difference and the terms of this integrated.

Before proceeding with the work of this paragraph the reader would do well to study again pp. 273 to 286, Part I.

Example 3.—Find the value of $\int 4 \sin 5t.3 \cos 2t dt$.

$$\begin{aligned} 4 \sin 5t.3 \cos 2t &= 12 \sin 5t \cos 2t \\ &= 6 \times 2 \sin 5t \cos 2t \\ &= 6\{\sin 7t + \sin 3t\} \qquad \text{(cf. p. 286, Part I).} \end{aligned}$$

Hence—

$$\begin{aligned} \int 4 \sin 5t.3 \cos 2t dt &= 6\left[\int \sin 7t dt + \int \sin 3t dt\right]. \\ &= 6\left[-\frac{1}{7}\cos 7t - \frac{1}{3}\cos 3t + C\right] \\ &= \underline{6C - \frac{6}{7}\cos 7t - 2\cos 3t.} \end{aligned}$$

Example 4.—Find $\int \sin^2 x dx$.

$$\begin{aligned} \cos 2x &= 1 - 2\sin^2 x, \text{ so that—} \\ \sin^2 x &= \frac{1 - \cos 2x}{2} \qquad \text{(cf. p. 280, Part I)} \end{aligned}$$

$$\begin{aligned} \therefore \quad \int \sin^2 x\, dx &= \frac{1}{2}\left[\int 1 dx - \int \cos 2x\, dx\right] \\ &= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x + C\right] \\ &= \underline{\cdot 5x - \cdot 25 \sin 2x + \cdot 5C.} \end{aligned}$$

Example 5.—Find $\int \tan^2 x dx$.

We know that $\sec^2 x = 1 + \tan^2 x$.

$$\begin{aligned} \therefore \quad \int \tan^2 x dx = \int (\sec^2 x - 1)\, dx &= \int \sec^2 x dx - \int 1\, dx. \\ &= \underline{\tan x - x + C.} \end{aligned}$$

Integration by Substitution.—At times a substitution aids the integration, but the cases in which this happens can only be distinguished after one has become perfectly familiar with the different types.

$\int \frac{(2ax+b)dx}{ax^2+bx+c}$ is a type to which this method applies.

In this fraction it will be observed that the numerator is exactly the differential of the denominator. Hence if u be written for the

denominator, the numerator may be replaced by du, so that the integral reduces to the simple form $\int \frac{du}{u}$, *i. e.*, $\log u + \log C$.

$$\begin{aligned}
\text{For if} \qquad & u = ax^2+bx+c \\
& \frac{du}{dx} = 2ax+b \qquad du = \frac{du}{dx} \cdot dx \\
\text{or} \qquad & du = (2ax+b)dx. \\
\text{Hence} \qquad & \int \frac{(2ax+b)dx}{ax^2+bx+c} = \int \frac{du}{u} \\
& = \log u + \log C \\
& = \log Cu \\
& = \underline{\log C(ax^2+bx+c)}.
\end{aligned}$$

In many cases integration may be effected by substitution of trigonometric for the algebraic functions; and Examples 6 to 10 illustrate this method of procedure.

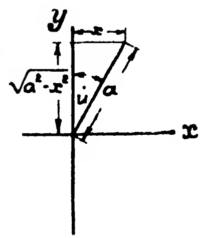


FIG. 36.

Example 6.—To find $\int \sqrt{a^2-x^2}\, dx$.

$$\begin{aligned}
\text{Let} \qquad & x = a \sin u, \text{ as illustrated by Fig. 36} \\
\text{then} \qquad & a^2-x^2 = a^2 - a^2 \sin^2 u = a^2(1-\sin^2 u) = a^2 \cos^2 u \\
\text{and} \qquad & \sqrt{a^2-x^2} = a \cos u, \text{ as will be seen from the figure.} \\
\text{Also} \qquad & \frac{dx}{du} = \frac{d(a \sin u)}{du} = a \cos u \\
\textit{i. e.,} \qquad & dx = a \cos u \, . \, du. \\
\therefore \qquad & \int \sqrt{a^2-x^2}\, dx = \int a \cos u \, . \, a \cos u \, du \\
& = a^2 \int \cos^2 u \, du \\
& = \frac{a^2}{2} \int (1+\cos 2u) du, \quad \text{since } \cos 2A = 2\cos^2 A - 1 \\
& = \frac{a^2}{2} \left(u + \frac{1}{2} \sin 2u + C \right).
\end{aligned}$$

Although this result is not expressed in terms of x, it is left in a form convenient for many purposes.

To express the result in terms of x—

$$\sin u = \frac{x}{a}, \text{ so that } u = \sin^{-1} \frac{x}{a}$$

$$\text{and also} \qquad \cos u = \sqrt{\frac{a^2-x^2}{a^2}}.$$

Hence $\qquad \frac{1}{2}\sin 2u = \sin u \cos u = \frac{x}{a}\times\frac{1}{a}\sqrt{a^2-x^2}$

$$\therefore \quad \int\sqrt{a^2-x^2}\,dx = \left(\frac{a^2}{2}\times\sin^{-1}\frac{x}{a}\right)+\left(\frac{a^2}{2}\times\frac{x}{a^2}\sqrt{a^2-x^2}\right)+K$$

$$= \underline{\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+K.}$$

Example 7.—To find $\int\frac{dx}{\sqrt{a^2-x^2}}$

Let $\qquad x = a\sin u \qquad$ *i. e.*, $u = \sin^{-1}\frac{x}{a}$

then $\qquad \frac{dx}{du} = \frac{d(a\sin u)}{du} = a\cos u$

also $\qquad \sqrt{a^2-x^2} = a\cos u, \qquad$ as before.

$$\therefore \quad \int\frac{dx}{\sqrt{a^2-x^2}} = \int\frac{a\cos u.du}{a\cos u} = \int 1\,du$$

$$= u+C$$

$$= \underline{\sin^{-1}\frac{x}{a}+C.}$$

Example 8.—To find $\int\frac{dx}{\sqrt{a^2+x^2}}$

In this case let $x = a\sinh u$, *i. e.*, $u = \sinh^{-1}\frac{x}{a}$

then $\qquad \frac{dx}{du} = \frac{d}{du}(a\sinh u) = a\cosh u.$

Now $\qquad \cosh^2 u-\sinh^2 u = 1 \qquad$ (cf. p. 291, Part I)

and thus $\qquad \cosh^2 u = 1+\sinh^2 u$

$$= 1+\frac{x^2}{a^2}$$

$$= \frac{a^2+x^2}{a^2}$$

or $\qquad a^2+x^2 = a^2\cosh^2 u$

and $\qquad \sqrt{a^2+x^2} = a\cosh u.$

$$\therefore \quad \int\frac{dx}{\sqrt{a^2+x^2}} = \int\frac{a\cosh u.du}{a\cosh u} = \int 1du = u+C$$

$$= \sinh^{-1}\frac{x}{a}+C.$$

Referring to p. 298, Part I, we see that—

$$\cosh^{-1}\frac{x}{a} = \log\left\{\frac{x+\sqrt{x^2-a^2}}{a}\right\}$$

and also $\qquad \sinh^{-1}\frac{x}{a} = \log\left\{\frac{x+\sqrt{x^2+a^2}}{a}\right\}.$

$$\therefore \int\frac{dx}{\sqrt{a^2+x^2}} = \underline{\sinh^{-1}\frac{x}{a}+C \text{ or } \log\left\{\frac{x+\sqrt{x^2+a^2}}{a}\right\}+C}$$

Example 9.—Find the value of $\int \frac{dx}{\sqrt{x^2-a^2}}$ and thence the value of $\int \frac{dx}{\sqrt{(x+a)^2-b^2}}$.

Dealing with the first of these—

let $\quad x = a\cosh u, \quad$ *i. e.*, $\quad u = \cosh^{-1}\frac{x}{a}$

$$\frac{dx}{du} = a\sinh u$$

or $\quad dx = a\sinh u\, du$

and $\quad x^2-a^2 = a^2\cosh^2 u - a^2 = a^2(\cosh^2 u - 1) = a^2\sinh^2 u$

$$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{dx}{a\sinh u} = \int \frac{a\sinh u\, du}{a\sinh u}$$

$$= \int du$$

$$= u + C$$

$$= \cosh^{-1}\frac{x}{a} + C$$

$$= \underline{\log\left\{\frac{x+\sqrt{x^2-a^2}}{a}\right\} + C.}$$

To evaluate the second integral, let $x+a = X$

then $dx = dX$ and $\int \frac{dx}{\sqrt{(x+a)^2-b^2}} = \int \frac{dX}{\sqrt{X^2-b^2}}$

$$= \log\left\{\frac{X+\sqrt{X^2-b^2}}{b}\right\} + C$$

$$= \underline{\log\left\{\frac{x+a+\sqrt{(x+a)^2-b^2}}{b}\right\} + C.}$$

Example 10.—Find the value of $\int \frac{dx}{a^2+x^2}$

The substitution in this case is—

$$a\tan u \text{ for } x$$

i. e., $\quad x = a\tan u \quad$ or $\quad u = \tan^{-1}\frac{x}{a}$

then $\quad \frac{dx}{du} = \frac{d}{dx}(a\tan u) = a\sec^2 u$

and $\quad x^2+a^2 = a^2\tan^2 u + a^2 = a^2(1+\tan^2 u) = a^2\sec^2 u$

$$\therefore \int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2 u\, du}{a^2\sec^2 u} = \frac{1}{a}\int du$$

$$= \frac{1}{a}u + C$$

$$= \underline{\frac{1}{a}\tan^{-1}\frac{x}{a} + C.}$$

By an extension of this result such an integral as $\int \frac{dx}{(x+a)^2+b^2}$ may be evaluated; for let $x+a = X$

$$\text{then} \qquad \frac{dX}{dx} = \frac{d(x+a)}{dx} = 1$$

$$\text{i. e.,} \qquad dX = dx$$

$$\text{hence} \qquad \int \frac{dx}{(x+a)^2+b^2} = \int \frac{dX}{X^2+b^2} = \frac{1}{b} \tan^{-1} \frac{X}{b} + C$$

$$\text{i. e.,} \quad \underline{\frac{1}{b} \tan^{-1} \frac{x+a}{b} + C.}$$

The following examples are illustrative of algebraic substitution or transformation.

Example 11.—To find the value of $\int \frac{dx}{\sqrt{2ax-x^2}}$.

Our plan in this case is so to arrange the integral that the method of a previous example may be applied.

$$\begin{aligned} 2ax-x^2 &= a^2-x^2-a^2+2ax \\ &= a^2-(x-a)^2. \end{aligned}$$

$$\text{Hence} \qquad \int \frac{dx}{\sqrt{2ax-x^2}} = \int \frac{dx}{\sqrt{a^2-(x-a)^2}} = \int \frac{dX}{\sqrt{a^2-X^2}}$$

the change from dx to dX being legitimate, since $\frac{dX}{dx} = \frac{d(x-a)}{dx} = 1$ and by *Example* 7, p. 150, the value of this integral is seen to be—

$$\sin^{-1} \frac{X}{a} + C.$$

$$\therefore \quad \int \frac{dx}{\sqrt{2ax-x^2}} = \underline{\sin^{-1} \frac{x-a}{a} + C.}$$

Example 12.—To find $\int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}$.

In this case the substitution is entirely algebraic.

$$\text{Let} \qquad u = \frac{1}{x} \text{ then } \frac{du}{dx} = -\frac{1}{x^2}$$

$$\text{or} \qquad dx = -x^2 du.$$

$$\begin{aligned} \text{Then} \qquad (a^2+x^2)^{\frac{3}{2}} &= \left(a^2+\frac{1}{u^2}\right)^{\frac{3}{2}} = (a^2u^2+1)^{\frac{3}{2}} \times \frac{1}{u^3} \\ &= (a^2u^2+1)^{\frac{3}{2}} \times x^3. \end{aligned}$$

$$\begin{aligned} \therefore \quad \int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}} = \int \frac{-x^2 du}{(a^2u^2+1)^{\frac{3}{2}} \times x^3} &= -\int \frac{du}{x(a^2u^2+1)^{\frac{3}{2}}} \\ &= -\int \frac{u du}{(a^2u^2+1)^{\frac{3}{2}}}. \end{aligned}$$

To evaluate this integral we must introduce another substitution.

Let
$$y = a^2u^2+1$$
then
$$\frac{dy}{du} = 2a^2u$$
or
$$udu = \frac{1}{2a^2}dy.$$

Hence the integral $= -\frac{1}{2a^2}\int\frac{dy}{y^{\frac{3}{2}}} = -\frac{1}{2a^2}\times -2\times\frac{1}{y^{\frac{1}{2}}}+C$

$$= \frac{1}{a^2y^{\frac{1}{2}}}+C = \frac{1}{a^2(a^2u^2+1)^{\frac{1}{2}}}+C$$
$$= \left(\frac{1}{a^2}\times\frac{1}{\left(a^2\times\frac{1}{x^2}+1\right)^{\frac{1}{2}}}\right)+C$$
$$= \frac{x}{a^2(a^2+x^2)^{\frac{1}{2}}}+C.$$

Example 13.—The equation $\frac{dz}{dt} = \frac{At}{\sqrt{t^2-1}}$ occurs in the statement of the mathematical theory of fluid motion, which is of value in connection with aeroplane design. Solve the equation for z.

To obtain z from $\frac{dz}{dt}$ we must integrate with regard to t; and to effect the integration let $u = t^2-1$, so that $\frac{du}{dt} = \frac{d(t^2-1)}{dt} = 2t$

or
$$dt = \frac{du}{2t}.$$

Then
$$z = \int\frac{Atdt}{\sqrt{t^2-1}} = \int\frac{Atdu}{u^{\frac{1}{2}}2t} = \int\frac{Adu}{2u^{\frac{1}{2}}}$$
$$= A\left(\frac{1}{2}\times 2u^{\frac{1}{2}}\right)+C$$
$$= Au^{\frac{1}{2}}+C$$
$$\text{or}\quad A\sqrt{t^2-1}+C.$$

Many difficult integrals of the form $\int\frac{dx}{x(a+bx^n)}$ can be evaluated by the substitution $z = x^{-n}$.

For if $z = x^{-n}$
$$\log z = -n\log x$$
$$\frac{d\log z}{dz} = -n\frac{d\log x}{dz}$$
$$\frac{1}{z} = -n\frac{d\log x}{dx}\times\frac{dx}{dz}$$
$$\frac{1}{z} = -n\times\frac{1}{x}\times\frac{dx}{dz}$$
or
$$dx = -\frac{xdz}{nz}.$$

The integral $\int \frac{dx}{x(a+bx^n)}$ thus reduces to $-\frac{1}{n}\int \frac{dz}{(az+b)}$, the value of which is $-\frac{1}{na}\log(az+b)$ or $\frac{1}{na}\log\left(\frac{x^n}{a+bx^n}\right)$.

Example 14.—Evaluate $\int \frac{dx}{x(4+5x^7)}$.

For x^7 write z^{-1}, so that in comparison with the standard form $n = 7$.

Then $\int \frac{dx}{x(4+5x^7)} = -\frac{1}{7}\int \frac{dz}{4z+5} = -\frac{1}{28}\left\{\log(4z+5)+\log C\right\}$

$$= -\frac{1}{28}\left\{\log\left(\frac{4}{x^7}+5\right)+\log C\right\}$$

$$= -\frac{1}{28}\left\{\log\left(\frac{4+5x^7}{x^7}\right)+\log C\right\}$$

$$= -\frac{1}{28}\left\{\log\frac{C(4+5x^7)}{x^7}\right\}$$

$$= \frac{1}{28}\log\left(\frac{x^7}{C(4+5x^7)}\right).$$

Example 15.—Find the value of $\int \frac{x^4dx}{(1-2x)^{\frac{1}{2}}}$.

It will be observed that the denominator is a surd quantity; and in many such cases it is advisable to choose a substitution that rationalises the denominator. Thus in this case let $u^2 = 1-2x$.

Then— $\qquad \frac{du^2}{dx} = \frac{du^2}{du}\times\frac{du}{dx} = 2u\frac{du}{dx}$

and $\qquad \frac{d}{dx}(1-2x) = -2$

so that $\qquad 2u\frac{du}{dx} = -2 \quad$ or $\quad dx = -udu.$

Also $1-2x = u^2$, whence $\frac{1-u^2}{2} = x$ and $x^4 = \left(\frac{1-u^2}{2}\right)^4$.

Expanding by the Binomial Theorem—

$$x^4 = \frac{1}{16}(1-4u^2+6u^4-4u^6+u^8).$$

Hence $\int \frac{x^4dx}{(1-2x)^{\frac{1}{2}}}$

$$= \frac{1}{16}\int \frac{(1-4u^2+6u^4-4u^6+u^8)\times -udu}{u}$$

$$= -\frac{1}{16}\left(u-\frac{4u^3}{3}+\frac{6u^5}{5}-\frac{4u^7}{7}+\frac{u^9}{9}\right)+C$$

$$= -\frac{u}{16}\left(1-\frac{4}{3}u^2+\frac{6u^4}{5}-\frac{4u^6}{7}+\frac{u^8}{9}\right)+C$$

$$= -\frac{(1-2x)^{\frac{1}{2}}}{16}\left[1-\frac{4}{3}(1-2x)+\frac{6}{5}(1-2x)^2-\frac{4}{7}(1-2x)^3+\frac{1}{9}(1-2x)^4\right]+C$$

which result could be further simplified if desired.

The next example introduces the substitution of an algebraic for a trigonometric function.

Example 16.—To find the value of $\int \frac{dx}{\sin x}$

Since $\sin 2A = 2 \sin A \cos A$, then $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$.

Hence— $$\int \frac{dx}{\sin x} = \frac{1}{2}\int \frac{dx}{\sin \frac{x}{2} \cos \frac{x}{2}} = \frac{1}{2}\int \frac{dx}{\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \times \cos^2 \frac{x}{2}}$$

$$= \frac{1}{2}\int \frac{\sec^2 \frac{x}{2} dx}{\tan \frac{x}{2}}.$$

Now let $$u = \tan \frac{x}{2}$$

then $$\frac{du}{dx} = \frac{1}{2} \sec^2 \frac{x}{2}$$

or $$dx = \frac{2du}{\sec^2 \frac{x}{2}}$$

$\therefore$ $$\int \frac{dx}{\sin x} = \frac{1}{2}\int \frac{\sec^2 \frac{x}{2} . 2du}{\sec^2 \frac{x}{2} . u}$$

$$= \int \frac{du}{u} = \log u + C$$

$$= \log \tan \frac{x}{2} + C.$$

Integration by Parts.—When differentiating a product, use is made of the rule—

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$ {Refer p. 70.}

If this equation be integrated throughout, with respect to x—

$$uv = \int vdu + \int udv$$

or, transposing—

$$\int udv = uv - \int vdu.$$

Many products may be integrated by the use of this rule.

Example 17.—To find $\int 4x.e^x dx$.

Let $u = 4x$, *i. e.*, $du = 4dx$

and let $dv = e^x dx$, *i. e.*, $v = e^x$.

Then
$$\begin{aligned}\int 4x.e^x dx &= \int u dv \\ &= uv - \int v du \\ &= 4x.e^x - \int e^x.4dx \\ &= \underline{4x.e^x - 4e^x + C.}\end{aligned}$$

Example 18.—Find $\int 5x^2.e^{4x}dx$.

Let $u = 5x^2$, *i. e.*, $du = 10x dx$

and $dv = e^{4x}dx$, *i. e.*, $v = \frac{1}{4}e^{4x}$.

Then
$$\begin{aligned}\int 5x^2.e^{4x}dx &= \int u dv \\ &= uv - \int v du \\ &= 5x^2 . \frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x}.10x dx \\ &= \frac{5}{4}x^2.e^{4x} - \frac{5}{2}\int x e^{4x} dx\end{aligned}$$

Now
$$\begin{aligned}\int x e^{4x} dx &= x.\frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x}dx \\ &= \frac{1}{4}x.e^{4x} - \frac{1}{16}e^{4x}\end{aligned} \qquad \left[\begin{array}{l}\text{where } u = x \\ \text{and } v = \frac{1}{4}e^{4x}\end{array}\right.$$

$\therefore$
$$\begin{aligned}\int 5x^2 e^{4x} dx &= \frac{5}{4}x^2.e^{4x} - \frac{5}{2}\left(\frac{1}{4}x.e^{4x} - \frac{1}{16}e^{4x}\right) + C \\ &= \underline{\frac{5}{4}e^{4x}\left\{x^2 - \frac{x}{2} + \frac{1}{8}\right\} + C.}\end{aligned}$$

Example 19.—To find $\int e^{ax}\sin(bx+c)dx$ and also—

$$\int e^{ax}\cos(bx+c)dx.$$

[The two integrals must be worked together.]

Dealing with the first, which we shall denote by M—

Let $u = \sin(bx+c)$, then $du = b\cos(bx+c)dx$

and $dv = e^{ax}dx$, so that $v = \int e^{ax}dx = \frac{1}{a}e^{ax}$.

Then
$$\begin{aligned}M &= \frac{1}{a}e^{ax}\sin(bx+c) - \int \frac{1}{a}e^{ax}\,b\cos(bx+c)dx \\ &= \frac{1}{a}e^{ax}\sin(bx+c) - \frac{b}{a}\int e^{ax}\cos(bx+c)dx \\ &= \frac{1}{a}e^{ax}\sin(bx+c) - \frac{b}{a}N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)\end{aligned}$$

where N stands for the second integral whose value we are finding.

By developing this second integral along similar lines we arrive at the value—

$$N = \frac{1}{a}e^{ax}\cos(bx+c)+\frac{b}{a}M \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We have thus a pair of simultaneous equations to solve.

Multiplying (1) by b and (2) by a and transposing—

$$bM = \frac{b}{a}e^{ax}\sin(bx+c)-\frac{b^2}{a}N$$

$$bM = -e^{ax}\cos(bx+c)+aN.$$

Subtracting $\quad 0 = e^{ax}\left[\frac{b}{a}\sin(bx+c)+\cos(bx+c)\right]-N\left(\frac{b^2}{a}+a\right)$

whence $\quad N = e^{ax}\left[\frac{b\sin(bx+c)+a\cos(bx+c)}{a^2+b^2}\right]$

and, by substitution, $M = e^{ax}\left[\frac{a\sin(bx+c)-b\cos(bx+c)}{a^2+b^2}\right]$

$$\therefore \quad \int e^{ax}\sin(bx+c)dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c)-b\cos(bx+c)]+C$$

$$\text{and} \quad \int e^{ax}\cos(bx+c)dx = \frac{e^{ax}}{a^2+b^2}[b\sin(bx+c)+a\cos(bx+c)]+C.$$

Example 20.—An electric current i whose value at any time t is given by the relation $i = I\sin pt$ is passed through the two coils of a wattmeter; the resistances of the two coils being R_1 and R_2 respectively, and their respective inductances L_1 and L_2. Then to find the separate currents in the two branches it is necessary to evaluate the integral—

$$\int Qe^{Pt}dt \quad \text{where } P = \frac{R_1+R_2}{L_1+L_2}$$

$$\text{and} \qquad Q = \frac{R_1 I}{L_1+L_2}\sin pt+\frac{L_1 pI}{L_1+L_2}\cos pt.$$

Evaluate this integral.

$$Q = \frac{I}{L_1+L_2}(R_1\sin pt+pL_1\cos pt) = \frac{I}{L_1+L_2}\sqrt{R_1^2+p^2L_1^2}\sin(pt+c)$$

where $\quad c = \tan^{-1}\frac{pL_1}{R_1}$ (see Part I, p. 276)

or $\quad Q = M\sin(pt+c)$, where $M = \frac{I}{L_1+L_2}\sqrt{R_1^2+p^2L_1^2}$.

Then $\quad \int Qe^{Pt}dt = \int e^{Pt}M\sin(pt+c)dt = M\int e^{Pt}\sin(pt+c)dt$

and this integral is of the type just discussed; its value being—

$$\frac{Me^{Pt}}{P^2+p^2}[P\sin(pt+c)-p\cos(pt+c)]+C$$

and in this form it is convenient to leave it, since in any numerical application it would be an easy matter to evaluate P, M and c before substituting into this result.

Some miscellaneous examples now follow, involving the use of the methods of this chapter.

Example 21.—Find the value of $\int_1^2 \frac{5\,dx}{x^2+8x+15}$.

$$\text{Let} \qquad \frac{5}{x^2+8x+15} = \frac{A}{x+3}+\frac{B}{x+5} = \frac{A(x+5)+B(x+3)}{x^2+8x+15}$$

$$i.e., \qquad 5 = A(x+5)+B(x+3).$$

$$\text{Let} \quad x = -5, \text{ then } 5 = 0-2B$$

$$B = -2\cdot5.$$

$$\text{Let} \quad x = -3, \text{ then } 5 = 2A+0$$

$$A = 2\cdot5, \; i.e., \frac{5}{x^2+8x+15} = \frac{2\cdot5}{x+3}-\frac{2\cdot5}{x+5}.$$

$$\therefore \qquad \int_1^2 \frac{5\,dx}{x^2+8x+15} = \int_1^2 \frac{2\cdot5\,dx}{x+3} - \int_1^2 \frac{2\cdot5\,dx}{x+5}$$

$$= 2\cdot5\left[\log(x+3)-\log(x+5)\right]_1^2$$

$$= 2\cdot5\,[\log 5-\log 7-\log 4+\log 6]$$

$$= 2\cdot5\,[1\cdot6094-1\cdot9459-1\cdot3863+1\cdot7918]$$

$$= 2\cdot5\times\cdot069 = \underline{\cdot1724}.$$

As an alternative method of solution, the graphic process of integration possesses certain advantages in a case such as this.

It might even be advisable, in all cases of definite integrals where the algebraic integration involves rather difficult rules, to treat the question both algebraically and graphically, the latter method serving as a very good check on the accuracy of the former.

$$\text{In this example} \int_1^2 \frac{5\,dx}{x^2+8x+15} = 5\int_1^2 \frac{dx}{(x+3)(x+5)} = 5\int_1^2 y\,dx$$

$$\text{where} \qquad y = \frac{1}{(x+3)(x+5)}$$

hence it is necessary to plot the curve $y = \frac{1}{(x+3)(x+5)}$ and find the area between it, the axis of x and the ordinates through $x = 1$ and $x = 2$.

The table for the plotting reads—

x	1	1·2	1·4	1·6	1·8	2
y	·04167	·03841	·0355	·03294	·03064	·02857

and from these values the curve AB is plotted in Fig. 37.

The sum curve for AB is the curve CD, the last ordinate of which, measured according to the scale of area, is ·0095. This figure is the area between the curve AB, the ordinates through $x = 1$ and $x = 2$, and the base line through $y = \cdot 025$; and hence the full area under the curve AB = ·0095 + area of a rectangle ·025 by 1, *i. e.*, ·0095 + ·025 or ·0345.

Thus
$$\int_1^2 y\,dx, \text{ i. e., } \int_1^2 \frac{dx}{(x^2+8x+15)} = \cdot 0345$$
$$\int_1^2 \frac{5\,dx}{(x^2+8x+15)} = 5 \times \cdot 0345 = \underline{\cdot 1725.}$$

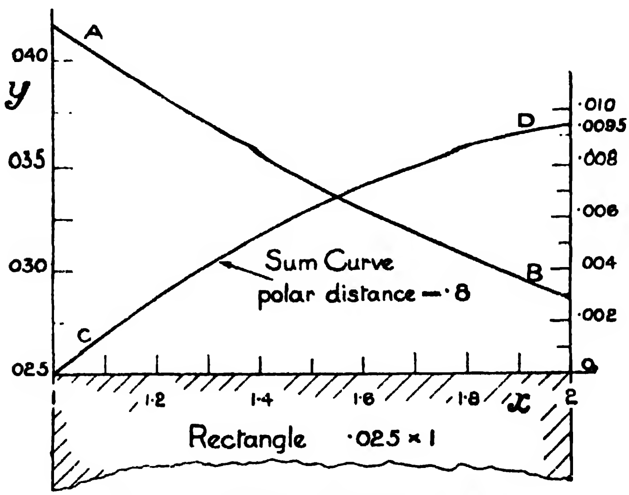


FIG. 37.—Graphic Integration of Ex. 21.

Example 22.—Find the value of $\int \frac{dx}{4x^2-9}$.

$$4x^2-9 = 4\left(x^2-\frac{9}{4}\right) = 4\left[x^2-\left(\frac{3}{2}\right)^2\right].$$

Now
$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log C\,\frac{(x-a)}{(x+a)} \quad \text{(see \textit{Example} 1, p. 147)}$$

$$\therefore \quad \int \frac{dx}{4x^2-9} = \frac{1}{4}\int \frac{dx}{x^2-\left(\frac{3}{2}\right)^2} = \frac{1}{2\times\frac{3}{2}} \times \frac{1}{4}\log \frac{C\left(x-\frac{3}{2}\right)}{\left(x+\frac{3}{2}\right)}$$
$$= \frac{1}{12}\log\frac{C(2x-3)}{2x+3}$$
$$\text{or } \underline{\log\left\{\frac{C(2x-3)}{(2x+3)}\right\}^{\frac{1}{12}}}.$$

Example 23.—Find the value of $\int_4^7 \frac{72x-20}{9x^2-5x+12}dx$.

In this case the numerator is of the first degree in x, whilst the denominator is of the second degree. Also we notice that the derivative of the denominator is $18x-5$, and the numerator is $4(18x-5)$. Thus the derivative of the denominator and the numerator are alike except as regards the constant factor 4. Hence the substitution will be u for $9x^2-5x+12$.

If $u = 9x^2-5x+12$, $\frac{du}{dx} = 18x-5$ or $du = (18x-5)dx$

so that $(72x-20)dx = 4(18x-5)dx = 4du$

$$\therefore \quad \int_4^7 \frac{72x-20}{9x^2-5x+12}dx = \int_{x=4}^{x=7} 4\frac{du}{u} = 4\int_{x=4}^{x=7}\frac{du}{u}$$
$$= 4\Big(\log_e u\Big)_{x=4}^{x=7}$$
$$= 4\Big[\log_e (9x^2-5x+12)\Big]_4^7$$
$$= 4\ (\log_e 418-\log_e 136)$$
$$= 4\ (\log_e 4{\cdot}18-\log_e 1{\cdot}36)$$
$$= 4\ (1{\cdot}4303-{\cdot}3075)$$
$$= \underline{4{\cdot}49.}$$

Example 24.—To find the value of $\int\frac{dx}{x^2+6x+15}$.

This is evidently of the type $\int\frac{dx}{(x+a)^2+b^2}$

for $\quad x^2+6x+15 = x^2+6x+9+6$
$$= (x+3)^2+(\sqrt{6})^2$$

so that $\quad a = 3$ and $b = \sqrt{6}$.

$$\therefore \quad \int\frac{dx}{x^2+6x+15} = \frac{1}{b}\tan^{-1}\frac{x+a}{b}+C \quad \text{(cf. \textit{Example} 10, p. 151)}$$
$$= \underline{\frac{1}{\sqrt{6}}\tan^{-1}\frac{x+3}{\sqrt{6}}+C.}$$

Example 25.—For a single straight wire at a potential different from that of the earth, if r = radius of wire in cms., l = length of wire in cms., σ = surface density of charge in electrostatic units per sq. cm., then the potential P at any point on the axis of the wire due to the charge on a length δx is given by—

$$P = \frac{2\pi r\sigma\delta x}{\sqrt{r^2+x^2}}$$

so that the potential at the middle point $= 2\pi r\sigma\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{dx}{\sqrt{r^2+x^2}}$.

Evaluate this integral.

This integral is of the type $\int\frac{dx}{\sqrt{a^2+x^2}}$.

$$\therefore \int\frac{dx}{\sqrt{r^2+x^2}} = \log\left\{\frac{x+\sqrt{x^2+r^2}}{r}\right\} \quad \text{(cf. \textit{Example} 8, p. 150)}$$

$$\therefore \qquad P_m = 2\pi r\sigma\left[\log\left\{\frac{x+\sqrt{x^2+r^2}}{r}\right\}\right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$= 2\pi r\sigma\left[\log\left\{\frac{\frac{l}{2}+\sqrt{\frac{l^2}{4}+r^2}}{r}\right\}-\log\left\{\frac{-\frac{l}{2}+\sqrt{\frac{l^2}{4}+r^2}}{r}\right\}\right]$$

$$= 2\pi r\sigma\log\left\{\frac{\sqrt{\frac{l^2}{4}+r^2}+\frac{l}{2}}{\sqrt{\frac{l^2}{4}+r^2}-\frac{l}{2}}\right\}$$

$$= 2\pi r\sigma\log\left\{\frac{\sqrt{\frac{l^2}{4r^2}+1}+\frac{l}{2r}}{\sqrt{\frac{l^2}{4r^2}+1}-\frac{l}{2r}}\right\}$$

or $$\underline{\pi d\sigma\log\left\{\frac{\sqrt{\left(\frac{l}{d}\right)^2+1}+\frac{l}{d}}{\sqrt{\left(\frac{l}{d}\right)^2+1}-\frac{l}{d}}\right\}} \text{ where } d \text{ is the diam. of wire.}$$

The following example involves the use of three of the methods of this chapter.

Example 26.—Find the value of $\int\frac{(5x+4)dx}{(x^2+2x+7)(x+3)}$.

The fraction under the integral sign should first be resolved into partial fractions.

Let $$\frac{5x+4}{(x^2+2x+7)(x+3)} = \frac{Ax+B}{(x^2+2x+7)}+\frac{C}{(x+3)} \quad \text{(cf. Part I, p. 454)}$$

$$= \frac{(Ax+B)(x+3)+C(x^2+2x+7)}{(x^2+2x+7)(x+3)}$$

i. e., $$5x+4 = (Ax+B)(x+3)+C(x^2+2x+7).$$

Let $x = -3$, then—

$$-11 = C(9-6+7) = 10C$$

i. e., $$C = -\frac{11}{10}.$$

Values of A and B can be found by equating coefficients of x and also those of x^2.

By equating coefficients of x^2, $0 = A+C$ and hence $A = \frac{11}{10}$.

By equating coefficients of x, $5 = 3A+B+2C = \frac{33}{10}+B-\frac{22}{10}$.

$$\therefore \quad B = \frac{39}{10}.$$

Hence the fraction—
$$= \frac{\frac{11}{10}x+\frac{39}{10}}{x^2+2x+7}+\frac{\left(-\frac{11}{10}\right)}{x+3}$$
$$= \frac{1}{10}\left\{\frac{11x+39}{x^2+2x+7}-\frac{11}{x+3}\right\}.$$

We can make the numerator of the first of these fractions into some multiple of the derivative of the denominator; thus—

The derivative of the denominator—

$$\begin{aligned} &= 2x+2 = 2(x+1) \\ \text{and the numerator } &= 11x+11+28 = 11(x+1)+28 \\ \text{and if } \quad u &= x^2+2x+7 \\ \text{then } \quad du &= 2(x+1)dx \\ \text{and } \{11(x+1)+28\}dx &= 11(x+1)dx+28dx \\ &= \frac{11}{2}du+28dx \end{aligned}$$

$$\therefore \quad \int\frac{(5x+4)dx}{(x^2+2x+7)(x+3)}$$
$$= \frac{1}{10}\int\left\{\frac{11x+39}{x^2+2x+7}-\frac{11}{x+3}\right\}dx$$
$$= \frac{1}{10}\left\{\int\frac{(11x+39)dx}{x^2+2x+7}-\int\frac{11dx}{x+3}\right\}$$
$$= \frac{1}{10}\left\{\int\frac{11}{2u}du+\int\frac{28dx}{x^2+2x+7}-\int\frac{11dx}{x+3}\right\}$$
$$= \frac{1}{10}\left\{\frac{11}{2}\log u+\frac{28}{\sqrt{6}}\tan^{-1}\frac{(x+1)}{\sqrt{6}}-11\log(x+3)+C\right\} \quad \left[\begin{aligned} &x^2+2x+7 \\ &= x^2+2x+1+6 \\ &= (x+1)^2+(\sqrt{6})^2 \end{aligned}\right]$$
$$= \frac{11}{20}\log(x^2+2x+7)+\frac{14}{5\sqrt{6}}\tan^{-1}\frac{(x+1)}{\sqrt{6}}-\frac{11}{10}\log(x+3)+C$$
$$\text{or } \log\frac{(x^2+2x+7)^{\frac{11}{20}}}{(x+3)^{\frac{11}{10}}}+\frac{14}{5\sqrt{6}}\tan^{-1}\frac{(x+1)}{\sqrt{6}}+C.$$

Example 27.—An integral required in the discussion of probability is $\int_0^\infty e^{-x^2}dx$. Find a value for this.

Let this integral be denoted by I, *i. e.*, $I = \int_0^\infty e^{-x^2}dx$.

Replace x by ax and thus dx by adx.

Then $I = \int_0^\infty e^{-a^2x^2}adx$.

Multiply all through by e^{-a^2}.

Then $Ie^{-a^2} = \int_0^\infty e^{-a^2(1+x^2)}.adx$ and integrate throughout with regard to a; thus—

$$\int_0^\infty Ie^{-a^2}da = \int_{x=0}^{x=\infty}\int_{a=0}^{a=\infty} e^{-a^2(1+x^2)}adadx.$$

But $\int_0^\infty Ie^{-a^2}da = I\int_0^\infty e^{-a^2}da = I\times I$ since $\int_0^\infty e^{-x^2}dx$ and $\int_0^\infty e^{-a^2}da$ have the same value,

$$\text{hence} \qquad I^2 = \int_{x=0}^{x=\infty}\int_{a=0}^{a=\infty} e^{-a^2(1+x^2)}adadx \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The value of the double integral on the right-hand side will be found by integrating first with regard to a and then with regard to x.

Dealing with the "inner" integral—

$$\int_{a=0}^{a=\infty} e^{-a^2(1+x^2)}ada$$

let $A = a^2$, then $dA = 2ada$, and let M represent $(1+x^2)$.

$$\text{Then} \int_{a=0}^{a=\infty} e^{-a^2(1+x^2)}.ada = \int_{a=0}^{a=\infty} e^{-AM}.\frac{a}{2a}dA$$

$$= -\frac{1}{2M}\left(e^{-AM}\right)_0^\infty \quad \text{since the limits for A are 0 and } \infty \text{ also}$$

$$= -\frac{1}{2M}(e^{-\infty}-e^0)$$

$$= -\frac{1}{2M}(0-1) = \frac{1}{2M}.$$

Referring to equation (1) and substituting this value therein—

$$I^2 = \int_0^\infty \frac{1}{2M}dx = \int_0^\infty \frac{dx}{2(1+x^2)}$$

$$= \frac{1}{2}\left(\tan^{-1}x\right)_0^\infty \quad (\text{cf. } \textit{Example } 10, \text{ p. } 151)$$

$$= \frac{1}{2}(\tan^{-1}\infty - \tan^{-1}0)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4}.$$

$$\therefore \quad I = \frac{1}{2}\sqrt{\pi}.$$

As an extension of this result it could be proved that—

$$\int_0^\infty e^{-\frac{x^2}{h^2}}dx = \frac{h}{2}\sqrt{\pi}.$$

Reduction Formulæ.—Many of the exceedingly difficult integrals which arise in advanced problems of thermodynamics,

theory of stresses, and electricity may be made by suitable substitutions to depend upon standard results obtained by a process of reduction. To grasp thoroughly the underlying principles on which the process is based, it is well to commence with a study of the simpler types.

We desire to evaluate the integrals $\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$, $\int_0^{\frac{\pi}{2}} \cos^n\theta \, d\theta$ and $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta \, d\theta$, where m and n have any positive integral values.

Taking the case in which $n = m = 0$, we have the results reducing to the form $\int_0^{\frac{\pi}{2}} 1 d\theta$, the value of which we know to be $\frac{\pi}{2}$ . . . . (1)

If $m = n = 1$

$$\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = -\Big(\cos\theta\Big)_0^{\frac{\pi}{2}} = -(\cos 90^\circ - \cos 0^\circ) = 1 \quad . \quad . \quad (2)$$

$$\int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = \Big(\sin\theta\Big)_0^{\frac{\pi}{2}} = (\sin 90^\circ - \sin 0^\circ) = 1 \quad . \quad . \quad . \quad (3)$$

from which pair of results we may say—

$$\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{2} - \theta\right) d\theta$$

or more generally—

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx \quad . \quad . \quad . \quad . \quad . \quad (4)$$

a result of great usefulness.

$$\text{Also} \quad \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta d\theta = -\frac{1}{2\times 2}\Big(\cos 2\theta\Big)_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{4}(\cos\pi - \cos 0)$$

$$= \frac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

By the process of reduction of powers we may express the integral to be evaluated in such a way that it depends on results (1), (2), (3) or (5).

Thus—

$$\int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2\theta) d\theta = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} 1 d\theta - \int_0^{\frac{\pi}{2}} \cos 2\theta \, d\theta\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2}-0\right)$$

$$= \frac{\pi}{4}$$

and from equation (4)—

$$\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta = \frac{\pi}{4}.$$

Now let $n = 3$, *i. e.*, we wish to evaluate $\int_0^{\frac{\pi}{2}} \sin^3\theta \, d\theta$.

Then $$\int_0^{\frac{\pi}{2}} \sin^3\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^2\theta . \sin\theta \, d\theta = \int_0^{\frac{\pi}{2}} (1-\cos^2\theta) \sin\theta \, d\theta$$

$$= -\int_0^{\frac{\pi}{2}} (1-u^2) \, du$$

$\left(0 \text{ and } \frac{\pi}{2} \text{ being the limits for } \theta\right)$

u being written for $\cos\theta$ and $-du$ for $\sin\theta \, d\theta$, since $\frac{d\cos\theta}{d\theta} = -\sin\theta$

and thus $$\frac{du}{d\theta} = -\sin\theta.$$

Now $$\int_{\theta=0}^{\theta=\frac{\pi}{2}} (1-u^2) \, du = \left(u-\frac{u^3}{3}\right)_{\theta=0}^{\theta=\frac{\pi}{2}} = \left(\cos\theta - \frac{\cos^3\theta}{3}\right)_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{3}.$$

Thus $$\int_0^{\frac{\pi}{2}} \sin^3\theta \, d\theta = -\int_{\theta=0}^{\theta=\frac{\pi}{2}} (1-u^2) \, du = +\frac{2}{3}$$

and if n be written for 3 we note that the result may be expressed in the form $\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta = \frac{n-1}{n}$ and also $\int_0^{\frac{\pi}{2}} \cos^n\theta \, d\theta = \frac{n-1}{n}$, n being an odd integer.

Let $n = 4$, then $\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta$ is required.

Now $\sin^4\theta \, d\theta = \sin^3\theta . \sin\theta d\theta = u dv$, where $u = \sin^3\theta$,
and $dv = \sin\theta \, d\theta$ or $v = -\cos\theta$.

Hence $\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta = -\left(\cos\theta \sin^3\theta\right)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos\theta \,.\, d \sin^3\theta$

$$= -\left(\cos\theta \sin^3\theta\right)_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos\theta \,.\, 3 \sin^2\theta \cos\theta\, d\theta$$

$$= 0 + 3\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta\, d\theta$$

since $\left(\cos\theta \sin^3\theta\right)_0^{\frac{\pi}{2}} = (0\times 1)-(1\times 0) = 0.$

Now $\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^2\theta(1-\sin^2\theta)d\theta$

$$= \int_0^{\frac{\pi}{2}} (\sin^2\theta - \sin^4\theta)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta - \int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta$$

Hence— $\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta = 3\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta - 3\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta$

or $4\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta = 3\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$

and $\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta = \frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta.$

We have thus reduced the power by 2, and knowing the result for $\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$, we can finally state the value for $\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta$.

Thus $\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta = \frac{3}{4}\times\frac{\pi}{4} = \frac{3\pi}{16}$ or $\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$

or $\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta = \frac{(n-1)}{n}\times\frac{(n-3)}{(n-2)}\,\frac{\pi}{2}$, n being an even integer.

In like manner it could be shown that—

$$\int_0^{\frac{\pi}{2}} \sin^5\theta\, d\theta = \frac{5-1}{5}\int_0^{\frac{\pi}{2}} \sin^3\theta\, d\theta.$$

Thus $\int_0^{\frac{\pi}{2}} \sin^5\theta\, d\theta = \frac{5-1}{5}\times\frac{5-3}{5-2}$ or $\frac{4\cdot 2}{5\cdot 3}$

which is of the form $\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta = \frac{n-1}{n} \cdot \frac{n-3}{n-2}$

n being an odd integer.

Similarly—

$$\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta = \frac{6-1}{6}\int_0^{\frac{\pi}{2}} \sin^4\theta d\theta = \frac{6-1}{6} \times \frac{6-3}{6-2} \times \frac{6-5}{6-4} \times \frac{\pi}{2}$$

$$\text{or} \quad \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \times \frac{\pi}{2}$$

which is of the form—

$$\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta = \frac{(n-1)(n-3)(n-5)}{n(n-2)(n-4)} \frac{\pi}{2}$$

n being an *even* integer.

Summarising our results—

$$\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta = \int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = \frac{(n-1)(n-3)(n-5)\ldots 1}{n(n-2)(n-4)\ldots 2} \frac{\pi}{2}$$

if n is an *even* integer.

$$\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta = \int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = \frac{(n-1)(n-3)(n-5)\ldots 2}{n(n-2)(n-4)\ldots 1}$$

if n is an *odd* integer.

Example 28.—Find the value of $\int_0^{\frac{\pi}{2}} \sin^9 \theta d\theta$.

In this case $n = 9$ and is odd.

$$\text{Hence—} \quad \int_0^{\frac{\pi}{2}} \sin^9\theta\, d\theta = \frac{8 \cdot 6 \cdot 4 \cdot 2}{9 \cdot 7 \cdot 5 \cdot 3} = \underline{\frac{128}{315}}$$

Example 29.—Evaluate $\int_0^{\frac{\pi}{2}} \cos^{10}\theta\, d\theta$.

Here $n = 10$ and is even.

$$\text{Hence} \quad \int_0^{\frac{\pi}{2}} \cos^{10}\theta\, d\theta = \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \underline{\frac{63\pi}{512}}.$$

Example 30.—The expression $-\frac{4\rho l}{\pi+4}\int_\infty^1 \frac{\sqrt{t^2-1}}{t^3} dt$ gives the theoretical thrust on a plane moving through air. Evaluate this.

$$\sqrt{t^2-1} = \sqrt{t^2\left(1-\frac{1}{t^2}\right)} = t\sqrt{1-\sin^2 u} = t \cos u = \frac{\cos u}{\sin u}$$

if $\sin u$ is written in place of $\frac{1}{t}$.

Then since $\sin u = \frac{1}{t} = t^{-1}$ $\qquad \frac{d \sin u}{du} = \frac{dt^{-1}}{du} = \frac{dt^{-1}}{dt} \times \frac{dt}{du}$

or $$\cos u = -\frac{1}{t^2}\frac{dt}{du}$$

whence $$dt = -\frac{du \cos u}{\sin^2 u}.$$

Again, when $t = \infty$ $\qquad \frac{1}{t} = 0 \quad$ *i. e.*, $\sin u = 0 \quad$ or $\quad u = 0$

and $\quad$ when $t = 1$ $\qquad \frac{1}{t} = 1 \quad$ *i. e.*, $\sin u = 1 \quad$ or $\quad u = \frac{\pi}{2}$

Hence—

$$\int_\infty^1 \frac{\sqrt{t^2-1}}{t^3}dt = -\int_0^{\frac{\pi}{2}} \frac{\cos u \sin^2 u \cos u du}{\sin u \sin^2 u} = -\int_0^{\frac{\pi}{2}} \cos^2 u du = -\frac{\pi}{4}.$$

Thus $$-\frac{4pl}{\pi+4}\int_\infty^1 \frac{\sqrt{t^2-1}}{t^3}dt = -\frac{4pl}{\pi+4} \times -\frac{\pi}{4} = \underline{\frac{\pi pl}{\pi+4}}.$$

We can now direct our attention to the determination of the value of $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta$, where m and n are positive integers.

It is convenient to discuss a simple case first, viz.—

$$\int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta\, d\theta.$$

Let $\qquad u = \sin^2\theta \qquad$ so that $\qquad \frac{du}{d\theta} = \frac{d \sin^2\theta}{d \sin\theta} \times \frac{d \sin\theta}{d\theta}$

$$= 2 \sin\theta \cos\theta$$

and $$du = 2 \sin\theta \cos\theta\, d\theta$$

also let $\quad dv = \cos\theta\, d\theta \quad$ so that $\quad v = \sin\theta.$

Then, by integrating by parts—

$$\int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta\, d\theta = \left(\sin^3\theta\right)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2 \sin^2\theta \cos\theta\, d\theta$$

whence $$3\int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta\, d\theta = \left(\sin^3\theta\right)_0^{\frac{\pi}{2}} = 1$$

or $$\int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta\, d\theta = \frac{1}{3}$$

and $\quad \frac{1}{3}$ might be written as $\frac{m-1}{m+n}$.

Example 31.—H, the horizontal thrust on a circular arched rib carrying a uniformly distributed load w per foot run of the arch, is obtained from—

$$H = \frac{wR^4 \int_0^{\frac{\pi}{6}} \left(\frac{1}{4} - \sin^2\theta\right)(\cos\theta - \cdot 866)\, d\theta}{2R^3 \int_0^{\frac{\pi}{6}} (\cos\theta - \cdot 866)\, d\theta}$$

if the span is equal to the radius of curvature (see Fig. 38).

If $w = \cdot 5$ ton per foot, and the span $=$ 60 ft., find the value of H.

Here $w = \cdot 5$, R $=$ span $=$ 60.

Hence $$H = \frac{\cdot 5 \times 60}{2} \frac{\int_0^{\frac{\pi}{6}} \left(\frac{1}{4} - \sin^2\theta\right)(\cos\theta - \cdot 866)\, d\theta}{\int_0^{\frac{\pi}{6}} (\cos\theta - \cdot 866)\, d\theta}.$$

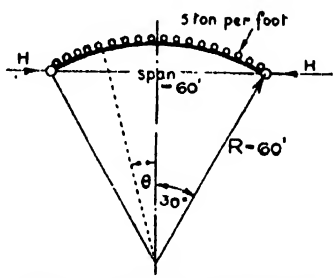


FIG. 38.—Circular Arched Rib.

Dealing with the numerator separately, as this alone presents any difficulty—

$$\left(\frac{1}{4} - \sin^2\theta\right)(\cos\theta - \cdot 866) = \frac{1}{4}\cos\theta - \cdot 2165 - \sin^2\theta\cos\theta + \cdot 866\sin^2\theta.$$

Then $\int\left(\frac{1}{4} - \sin^2\theta\right)(\cos\theta - \cdot 866)\, d\theta$

$$= \int\frac{1}{4}\cos\theta\, d\theta - \int\cdot 2165\, d\theta - \int\sin^2\theta\cos\theta\, d\theta + \int\cdot 866\sin^2\theta\, d\theta$$

but, as proved on p. 148, $\int\sin^2\theta\, d\theta = \frac{1}{2}\int 1 d\theta - \frac{1}{2}\int\cos 2\theta\, d\theta$

$$= \frac{\theta}{2} - \frac{1}{4}\sin 2\theta$$

and as proved on p. 168 $\int\sin^2\theta\cos\theta\, d\theta = \frac{1}{3}\sin^3\theta$

thus $\int_0^{\frac{\pi}{6}} \left(\frac{1}{4}-\sin^2\theta\right)(\cos\theta-\cdot866)\,d\theta$

$$= \left(\frac{1}{4}\sin\theta\right)_0^{\frac{\pi}{6}} - \left(\cdot2165\theta\right)_0^{\frac{\pi}{6}} - \frac{1}{3}\left(\sin^3\theta\right)_0^{\frac{\pi}{6}} + \cdot433\left(\theta-\frac{1}{2}\sin 2\theta\right)_0^{\frac{\pi}{6}}$$

$$= \left(\frac{1}{4}\times\cdot5-0\right) - \left(\frac{\cdot2165\times\pi}{6}-0\right) - \frac{1}{3}\left((\cdot5)^3-0\right) + \cdot433\left(\frac{\pi}{6}-\frac{1}{2}\times\cdot866-0+0\right)$$

$$= \cdot125-\cdot1134-\cdot0417+\cdot2267-\cdot1875 = \cdot0091.$$

Dealing now with the denominator—

$$\int_0^{\frac{\pi}{6}}(\cos\theta-\cdot866)\,d\theta = \left(\sin\theta-\cdot866\theta\right)_0^{\frac{\pi}{6}} = \cdot5-\cdot866\times\frac{\pi}{6} = \cdot5-\cdot4534 = \cdot0466.$$

Hence $\qquad H = \dfrac{15\times\cdot0091}{\cdot0466} = \underline{2\cdot93 \text{ tons.}}$

Carrying the investigation a step further, let us discuss the case of $\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta$.

$$\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta = \int_0^{\frac{\pi}{2}}(1-\cos^2\theta)\cos^2\theta\,d\theta = \int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta - \int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta$$

and from the previous result this value $= \dfrac{\pi}{4} - \dfrac{3\cdot1}{4\cdot2}\dfrac{\pi}{2}$

$$= \frac{1}{8}\frac{\pi}{2} \text{ or } \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

This result might be regarded as obtained by first reducing the power m by 2, and next that of $\cos^2\theta$ by 2.

Thus for the first step—

$$\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta = \frac{2-1}{4}\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

or

$$\frac{m-1}{m+n}\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

and for the second step—

$$\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta = \frac{2-1}{2}\int_0^{\frac{\pi}{2}}1\,d\theta$$

$$= \frac{2-1}{2}\cdot\frac{\pi}{2} \text{ or } \frac{n-1}{n+m}\frac{\pi}{2}$$

m for the second integral being zero.

In a similar fashion we might reduce $\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^3\theta\, d\theta$ as follows :—

First reduce m by 2, then—

$$\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^3\theta\, d\theta = \frac{4-1}{4+3}\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^3\theta\, d\theta.$$

Next reduce the power of $\sin^2\theta$ again by 2.

Thus $$\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^3\theta\, d\theta = \frac{4-1}{4+3}\cdot\frac{2-1}{2+3}\int_0^{\frac{\pi}{2}} \cos^3\theta\, d\theta.$$

Now reduce the power of $\cos^3\theta$ by 2, and remember that m is now $= 0$. Then—

$$\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^3\theta\, d\theta = \frac{4-1}{4+3}\cdot\frac{2-1}{2+3}\cdot\frac{3-1}{3+0}\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta = \frac{3\cdot 1\cdot 2}{7\cdot 5\cdot 3}.$$

In the evaluation of the integral $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta$ we thus proceed by reducing by 2 the powers of m and n in turn until they become 1 or 0. The various cases that arise are—

(*a*) m and n both even : in which case the final integral is

$$\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}.$$

(*b*) m and n both odd : in which case the final integral is

$$\int_0^{\frac{\pi}{2}} \sin\theta \cos\theta\, d\theta = \frac{1}{2}.$$

(*c*) m even and n odd or *vice-versa* : in which case the final integral is either $\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta$ or $\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta$, the value of either being 1.

The results for the three cases can be thus stated—

(*a*) m and n both even—

$$\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta = \frac{(m-1)\times(m-3)\times\ .\ .\ .\ 1}{(m+n)\times(m-2+n)\times\ .\ .\ .\ (n+2)}$$
$$\times\frac{(n-1)(n-3)\times\ .\ .\ .\ 1}{n(n-2)\times\ .\ .\ .\ 2}\times\frac{\pi}{2}$$

(*b*) m and n both odd—

$$\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta \, d\theta = \frac{(m-1)\times(m-3)\times \ldots 2}{(m+n)\times(m-2+n) \times \ldots (n+3)} \times \frac{(n-1)\times(n-3) \quad \ldots 2}{(n+1)\times(n-1)\times \ldots 4} \times \frac{1}{2}$$

since, after reducing the power of $\sin^m\theta$ by 2 at a time, we must be left with $\sin\theta \cos^n\theta$, so that the value of m to be used in the reduction of $\cos^n\theta$ must be taken as 1.

(*c*) m even and n odd—

$$\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta \, d\theta = \frac{(m-1)\times(m-3)\times \ldots 1}{(m+n)\times(m-2+n)\times \ldots (n+2)} \times \frac{(n-1)\times(n-3)\times \ldots 2}{n(n-2)\times \ldots 1} \times 1$$

Example 32.—Evaluate $\int_0^{\frac{\pi}{2}} \sin^8\theta \cos^{10}\theta \, d\theta$.

This is case (*a*), *i. e.*, with both m and n even.

Hence $$\int_0^{\frac{\pi}{2}} \sin^8\theta \cos^{10}\theta \, d\theta = \frac{7 \cdot 5 \cdot 3 \cdot 1}{18 \cdot 16 \cdot 14 \cdot 12} \times \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \times \frac{\pi}{2}$$

$$= \underline{\frac{35\pi}{131072}}.$$

Example 33.—Find the value of $\int_0^{\frac{\pi}{2}} \sin^3\theta \cos^5\theta \, d\theta$

This is case (*b*), *i. e.*, with both m and n odd.

Hence $$\int_0^{\frac{\pi}{2}} \sin^3\theta \cos^5\theta \, d\theta = \frac{2}{8} \times \frac{4 \cdot 2}{6 \cdot 4} \times \frac{1}{2} = \underline{\frac{1}{24}}.$$

Example 34.—Find the value of $\int_0^{\frac{\pi}{2}} \sin^7\theta \cos^4\theta \, d\theta$.

This is case (*c*), but with m odd and n even.

Hence $$\int_0^{\frac{\pi}{2}} \sin^7\theta \cos^4\theta \, d\theta = \frac{6 \cdot 4 \cdot 2}{11 \cdot 9 \cdot 7} \times \frac{3 \cdot 1}{5 \cdot 3} \times 1 = \underline{\frac{16}{1155}}.$$

The value of the foregoing formulæ is found in their employ-

ment in the evaluation of difficult algebraic functions, which may often be transformed by suitable trigonometric substitution.

Thus to evaluate $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, known as the First Eulerian Integral and usually denoted by the form $B(m, n)$, we may substitute $\sin^2\theta$ for x. Then, since when $x = 0$ $\sin^2\theta$ must $= 0$ and thus $\theta = 0$, and when $x = 1$ $\sin^2\theta$ must $= 1$ and thus $\theta = \frac{\pi}{2}$

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta(1-\sin^2\theta)^{n-1}dx$$

$$\left[\begin{aligned} \frac{dx}{d\theta} &= \frac{d\sin^2\theta}{d\theta} \\ &= \frac{d\sin^2\theta}{d\sin\theta}\cdot\frac{d\sin\theta}{d\theta} \\ &= 2\sin\theta\cos\theta \end{aligned}\right] \qquad \begin{aligned} &= \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{2n-2}\theta \times 2\sin\theta\cos\theta d\theta \\ &= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta \end{aligned}$$

and this can readily be evaluated.

Example 35.—Evaluate $\int_0^1 x^4(1-x^2)^{\frac{5}{2}}dx$.

Let $x = \sin\theta$ then $1-x^2 = 1-\sin^2\theta = \cos^2\theta$

and $\frac{dx}{d\theta} = \frac{d\sin\theta}{d\theta} = \cos\theta$.

Also when $x = 0$ $\sin\theta = 0$ and thus $\theta = 0$

and when $x = 1$ $\sin\theta = 1$,, ,, $\theta = \frac{\pi}{2}$.

Then—

$$\int_0^1 x^4(1-x^2)^{\frac{5}{2}}dx = \int_0^{\frac{\pi}{2}} \sin^4\theta\cos^5\theta\, d\theta\cos\theta = \int_0^{\frac{\pi}{2}} \sin^4\theta\cos^6\theta\, d\theta$$

$$= \frac{3\cdot 1}{10.8}\times\frac{5\cdot 3\cdot 1}{6.4.2}\times\frac{\pi}{2} = \frac{9\pi}{512}.$$

Another important result obtained by the process of reduction is the value of $\int_0^\infty e^{-x}x^n dx$. This is termed a Gamma Function; this particular integral being the $(n+1)$ Gamma Function denoted by $\Gamma(n+1)$.

Thus $$\int_0^\infty e^{-x}x^n dx = \Gamma(n+1)$$

and $$\int_0^\infty e^{-x}x^{n-1}dx = \Gamma(n)$$

the latter integral being also called the Second Eulerian Integral.

To evaluate $\Gamma(n+1)$ let $u = x^n$ and $dv = e^{-x}dx$
so that $v = -e^{-x}$ and $du = nx^{n-1}dx$.

Then $$\int_0^\infty e^{-x}x^n dx = \left(-e^{-x}x^n\right)_0^\infty - \int_0^\infty -e^{-x}\,.\,nx^{n-1}dx$$
$$= \left(-e^{-x}x^n\right)_0^\infty + n\int_0^\infty e^{-x}x^{n-1}dx.$$

Now when $x = \infty$ $e^{-x}x^n = \frac{1 \times \infty}{e^\infty}$, which can be proved $= 0$

and when $x = 0$ $e^{-x}x^n = \frac{1 \times 0}{e^0} = 0$.

Hence $$\int_0^\infty e^{-x}x^n dx = n\int_0^\infty e^{-x}x^{n-1}dx$$

or $$\Gamma(n+1) = n\Gamma(n).$$

In like manner it could be shown that—
$$\Gamma(n) = (n-1)\Gamma(n-1), \text{ and so on.}$$

If n is an integer it will be seen that we finally reduce to $\Gamma(1)$, *i. e.*, $\int_0^\infty e^{-x}dx$, the value of which is $-(e^{-\infty} - e^0) = -(0-1) = 1$.

Hence $$\Gamma(n+1) = n(n-1)(n-2) \ldots 1 = \underline{|n}.$$

This last relation will not hold, however, if n is not an integer, but the general method of attack holds good; and tables have been compiled giving the values of $\Gamma(n)$ for many values of n, whether n be an integer or fractional. Thus if an integral can be reduced to a Gamma function or a combination of Gamma functions, its complete evaluation may be effected by reference to the tables.

Example 36.—Evaluate the probability integral $\int_0^\infty e^{-\frac{x^2}{h^2}}dx$ by the aid of the Gamma function.

Let $X = \frac{x^2}{h^2}$ then $\frac{dX}{dx} = \frac{d\frac{x^2}{h^2}}{dx} = \frac{2x}{h^2}$

so that $dX = \frac{2xdx}{h^2}$

or $dx = \frac{h^2}{2}\frac{dX}{x} = \frac{h^2}{2}\frac{dX}{h\sqrt{X}} = \frac{hdX}{2\sqrt{X}}$.

Then $\int_0^\infty e^{-\frac{x^2}{h^2}}dx = \int_0^\infty e^{-X} \times \frac{hdX}{2\sqrt{X}} = \frac{h}{2}\int_0^\infty e^{-X}X^{-\frac{1}{2}}dX = \frac{h}{2} \times \Gamma\left(\frac{1}{2}\right)$

and the value of $\Gamma\left(\frac{1}{2}\right)$ is $\sqrt{\pi}$.

Hence $$\int_0^\infty e^{-\frac{x^2}{h^2}}dx = \frac{h}{2}\sqrt{\pi}.$$

On comparing with *Example* 27, p. 162, where a rather simpler form of the integral is evaluated, we see the great saving effected by the use of the Gamma function.

LIST OF INTEGRALS LIKELY TO BE OF SERVICE

$$\int(ax^n+b)\,dx = \frac{ax^{n+1}}{n+1}+bx+C.$$

$$\int ae^{bx}dx = \frac{a}{b}e^{bx}+C.$$

$$\int ba^{nx}dx = \frac{b}{n\log a}\cdot a^{nx}+C.$$

$$\int a\cos(bx+d)\,dx = \frac{a}{b}\sin(bx+d)+C.$$

$$\int a\sin(bx+d)\,dx = -\frac{a}{b}\cos(bx+d)+C.$$

$$\int a\tan(bx+d)\,dx = -\frac{a}{b}\log\cos(bx+d)+C.$$

$$\int a\cot(bx+d)\,dx = \frac{a}{b}\log\sin(bx+d)+C.$$

$$\int a\sec^2(bx+d)\,dx = \frac{a}{b}\tan(bx+d)+C.$$

$$\int a\operatorname{cosec}^2(bx+d)\,dx = -\frac{a}{b}\cot(bx+d)+C.$$

$$\int a\cosh(bx+d)\,dx = \frac{a}{b}\sinh(bx+d)+C.$$

$$\int a\sinh(bx+d)\,dx = \frac{a}{b}\cosh(bx+d)+C.$$

$$\int a\operatorname{sech}^2(bx+d)\,dx = \frac{a}{b}\tanh(bx+d)+C.$$

$$\int a\operatorname{cosech}^2(bx+d)\,dx = -\frac{a}{b}\coth(bx+d)+C.$$

$$\int e^{ax}\sin(bx+d)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+d)-b\cos(bx+d)]+C.$$

$$\int e^{ax}\cos(bx+d)\,dx = \frac{e^{ax}}{a^2+b^2}[b\sin(bx+d)+a\cos(bx+d)]+C.$$

$$\int \frac{dx}{\sin(bx+d)} = \frac{1}{b}\log\left(\tan\frac{bx+d}{2}\right)+C.$$

$$\int \frac{dx}{ax+b} = \frac{1}{a}\log(ax+b)+C.$$

$$\int \tan(bx+d)\,dx = -\frac{1}{b}\log\cos(bx+d)+C.$$

$$\int \cot(bx+d)\,dx = \frac{1}{b}\log\sin(bx+d)+C.$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C.$$

$$\int \frac{dx}{(x+a)^2+b^2} = \frac{1}{b}\tan^{-1}\frac{x+a}{b}+C.$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}+C.$$

$$\int \frac{dx}{(x+a)^2-b^2} = \frac{1}{2b}\log\frac{x+a-b}{x+a+b}+C.$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+C.$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)+C.$$

$$\int \frac{dx}{\sqrt{(x+a)^2+b^2}} = \log\left(\frac{x+a+\sqrt{(x+a)^2+b^2}}{b}\right)+C.$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)+C.$$

$$\int \frac{dx}{\sqrt{(x+a)^2-b^2}} = \log\left(\frac{x+a+\sqrt{(x+a)^2-b^2}}{b}\right)+C.$$

$$\int \frac{(ax+b)dx}{(ax^2+2bx+d)} = \frac{1}{2}\log(ax^2+2bx+d)+C.$$

$$\int \operatorname{cosec}(ax+b)\,dx = \frac{1}{a}\log\left(\tan\frac{ax+b}{2}\right)+C.$$

$$\int \sec(ax+b)\,dx = \frac{1}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax+b}{2}\right)+C.$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}+C.$$

$$\int \frac{dx}{\sqrt{2ax-x^2}} = \operatorname{vers}^{-1}\frac{x}{a}+C \quad \text{or} \quad 1-\cos^{-1}\frac{x}{a}+C.$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C.$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}+C.$$

$$\int\sqrt{(x+a)^2-b^2}\,dx = \frac{1}{2}(x+a)\sqrt{(x+a)^2-b^2}-\frac{b^2}{2}\cosh^{-1}\frac{x+a}{b}+C.$$

$$\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+C.$$

$$\int\sqrt{(x+a)^2+b^2}\,dx = \frac{1}{2}(x+a)\sqrt{(x+a)^2+b^2}+\frac{b^2}{2}\sinh^{-1}\frac{x+a}{b}+C.$$

$$\int\sin^2 x\,dx = \frac{x}{2}-\frac{1}{4}\sin 2x+C.$$

$$\int\cos^2 x\,dx = \frac{x}{2}+\frac{1}{4}\sin 2x+C.$$

$$\int\sin^m x\cos^n x\,dx = \frac{m-1}{m+n}\int\sin^{m-2}x\,.\cos^n x\,dx - \frac{\sin^{m-1}x\,.\cos^{n+1}x}{m+n}+C.$$

$$\int\frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2-x^2}}+C.$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

$$\int_0^\infty e^{-\frac{x^2}{h^2}}\,dx = \frac{h\sqrt{\pi}}{2}.$$

$$\int_0^{\frac{\pi}{2}}\sin^2\theta\,d\theta = \int_0^{\frac{\pi}{2}}\cos^2\theta\,d\,\theta = \frac{\pi}{4}.$$

$$\int_0^{\frac{\pi}{2}}\sin^n\theta\,d\theta = \int_0^{\frac{\pi}{2}}\cos^n\theta\,d\theta = \frac{(n-1)(n-3)(n-5)\ldots 1}{n(n-2)(n-4)\ldots 2}\,\frac{\pi}{2}$$

if n is an even integer.

$$\int_0^{\frac{\pi}{2}}\sin^n\theta\,d\theta = \int_0^{\frac{\pi}{2}}\cos^n\theta\,d\theta = \frac{(n-1)(n-3)(n-5)\ldots 2}{n(n-2)(n-4)\ldots 1}$$

if n is an odd integer.

$$\int_\infty^1\frac{\sqrt{x^2-1}}{x^3}\,dx = -\frac{\pi}{4}.$$

$$\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta\,d\theta = \frac{(m-1)(m-3)\ldots 1}{(m+n)(m+n-2)\ldots(n+2)}\times\frac{(n-1)(n-3)\ldots 1}{n(n-2)\ldots 2}\times\frac{\pi}{2}$$

if m and n are both even.

$$\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta\,d\theta = \frac{(m-1)(m-3)\ldots 2}{(m+n)(m+n-2)\ldots(n+3)}\times\frac{(n-1)(n-3)\ldots 2}{(n+1)(n-1)\ldots 4}\times\frac{1}{2}$$

if m and n are both odd.

$$\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta \, d\theta = \frac{(m-1)(m-3)\ldots 1}{(m+n)(m+n-2)\ldots(n+2)} \times \frac{(n-1)(n-3)\ldots 2}{n(n-2)\ldots 1}$$

if m is even and n is odd.

$$\int_{v_1}^{v_2} p\,dv \text{ (where } pv^n = C) = \frac{C}{1-n}(v_2^{1-n} - v_1^{1-n}) = \frac{p_2v_2 - p_1v_1}{1-n}.$$

$$\int_{v_1}^{v_2} p\,dv \text{ (where } pv = C) = C \log_e \frac{v_2}{v_1}.$$

$$\int_{\tau_1}^{\tau_2} \frac{a+b\tau}{\tau} \, d\tau = a \log \frac{\tau_2}{\tau_1} + b(\tau_2 - \tau_1).$$

$$\int \sin^{-1} ax \, dx = x \sin^{-1} ax + \frac{1}{a}\sqrt{1-a^2x^2} + C.$$

$$\int \tan^{-1} ax \, dx = x \tan^{-1} ax - \frac{1}{2a} \log(1+a^2x^2) + C.$$

$$\int (a^2-x^2)^{\frac{3}{2}} \, dx = \frac{x}{4}(a^2-x^2)^{\frac{3}{2}} + \frac{3a^2}{8}\left(x\sqrt{a^2-x^2} + a^2 \sin^{-1}\frac{x}{a}\right) + C.$$

$$\int_0^a (a^2-x^2)^{\frac{3}{2}} \, dx = \frac{3\pi a^4}{16}.$$

Exercises 15.—On Further Integration.

Evaluate the integrals in Nos. 1 to 18.

1. $\int \frac{dx}{2x^2-5}$. 2. $\int \frac{dx}{3x^2+6x+21}$. 3. $\int \frac{(x-1)dx}{9x^2-18x+17}$.

4. $\frac{4}{3}\int_1^{1\frac{1}{2}} \frac{(3{\cdot}5-y)dy}{y+3}$, which occurred when finding stresses in a crane hook.

5. $t = 95{\cdot}5\int_0^3 (6-h)^{\frac{1}{2}} dh$, referring to the time for emptying a tank.

6. $\int \sqrt{3-4y^2}\,dy$. 7. $\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin 5t \cos t \, dt$. 8. $\int 8 \cos^2 6t \, dt$.

9. $\int \frac{dx}{\sin^2 5x}$. 10. $\int_0^{\frac{\pi}{2}} \frac{WR}{EI}\left(\frac{1}{\pi} - \frac{1}{2}\sin\theta\right)R^2 \cos\theta \, d\theta$.

11. $\int 4 \tan 5t \, dt$. 12. $\int \sin^{-1} x \, dx$. 13. $\int x\sqrt{a^2-x^2}\,dx$.

14. $\int 5e^{3x} \sin 2x \, dx$. 15. $\int \frac{2dx}{(1+x^2)^2}$. 16. $\int_0^{\frac{\pi}{6}} x \sin x \, dx$.

17. $\int \cos\theta \sin^2\theta \, d\theta$. 18. $h = \int_0^l \frac{64fQ^2}{2g\pi^2} \frac{dx}{(d+kx)^5}$, relating to the flow of water through a pipe of uniformly varying diameter; the diameter at distance x from the small end being = small end diam. $+kx$.

19. The time taken (t secs.) to lower the level of the liquid in a certain vessel having two orifices in one side can be found from

$$dt = \frac{110dh}{\sqrt{h+12}+\sqrt{h}}$$

Find this time if the limits of h are 0 and 10.

20. Express e^{-t^2} as a series, and thence find the value of—

$$\frac{2}{\sqrt{\pi}}\cdot\int_0^t e^{-t^2}dt.$$

21. The maximum intensity of shearing stress over a circular section of radius $r = S = \frac{F}{2rI}\int_0^r 2(r^2-y^2)^{\frac{1}{2}}ydy.$

If $I = \frac{\pi r^4}{4}$, find a simple expression for S.

22. Evaluate the integral $\int\frac{dt}{t-\sqrt{t^2-4}}$. [*Hint.*—Rationalise the denominator.]

23. Write down the value of $\int_0^{\frac{\pi}{2}}\cos^8\theta d\theta.$

24. If the value of log $\Gamma(1{\cdot}85)$ is given in the tables as $\bar{1}{\cdot}9757$, find the value of $\int_0^\infty e^{-x}x^{1\cdot85}dx.$

25. Write down the value of $\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^7\theta d\theta.$

26. When finding the forces in a circular arched rib it was found necessary to find the value of—

$$2\int_0^{\frac{\pi}{6}}R^3\left(\cos^2\theta-\sqrt{3}\cos\theta+\frac{3}{4}\right)d\theta.$$ Evaluate this integral.

27. Evaluate the indefinite integral $\int\sqrt{21-5x-x^2}\,dx.$

28. The attraction F of a thin circular disc of radius r on a body on the axis of the disc, and distant z from the centre, is given by—

$$F = -2\pi\sigma kz\int_0^a\frac{rdr}{(r^2+z^2)^{\frac{3}{2}}}$$

where σ is the density of the disc, and k is a constant.
Find the value of F for this case.

29. Find the time t taken for the water in a conduit connecting a surge tank and a hydraulic turbine to change its velocity from V_1 to V_2; it being given that

$$t = \int_{V_1}^{V_2}\frac{LdV}{g(y-c(V^2-V_1^2))}$$

and $cs^2 = y + cV_1^2$.

CHAPTER VII

MEAN VALUES: ROOT MEAN SQUARE VALUES: VOLUMES: LENGTH OF ARC: AREA OF SURFACE OF SOLID OF REVOLUTION: CENTROID: MOMENT OF INERTIA

Determination of Mean Values.—It is frequently necessary to calculate the mean value of a varying quantity: thus if a variable force acts against a resistance, the work done will be dependent on the mean value of the force; or to take an illustration from electrical theory, if we can find the values of the current and electro-motive force at various instants during the passage of the current, then the mean rate of working is the mean value of their product.

The mean value of a series of values is found by adding the values together and then dividing by the number of values taken. If, however, a curve is drawn to give by its ordinate the magnitude of the quantity at any instant, the mean value of the quantity is determined by the mean height of the diagram, which is the area divided by the length of the base. This really amounts to the taking of an exceedingly large number of ordinates and then calculating their average.

The area may be measured by the planimeter, in which case the instrument may be set to record the mean height directly, or by any of the methods enumerated in Part I, Chap. VII.

A clear conception of the idea of mean values can be gained by consideration of the examples that follow; the first example being merely of an arithmetical nature.

Example 1.—The corresponding values of an electric current and the E.M.F. producing it are as in the table:—

C	0	1·8	3·5	5	6·1	6·8	6·1	5	3·5	1·8
E	0	9	17·5	25	30·5	34	30·5	25	17·5	9

Find the mean value of the power over the period during which these values were measured.

The power is measured by the product current × E.M.F., and thus the values of the power are—

0 16·2 61·25 125 186·05 231·2 186·05 125 61·25 16·2

the sum = 1008·2 the number of values = 10

hence the average or mean value $= \frac{1008 \cdot 2}{10} = \underline{100 \cdot 8}$.

A better result would probably be obtained if the values of the power were plotted and the area of the diagram found.

Thus in Fig. 39 the base is taken as 10 units (merely for convenience), and the area is found to be 1013 sq. units. Then the mean height of the diagram, which is $\frac{1013}{10} = 101 \cdot 3$, is also the mean value of the power; and a line drawn at a height of 101·3 units divides the figure in such a way that the area B is equal to the areas A+A.

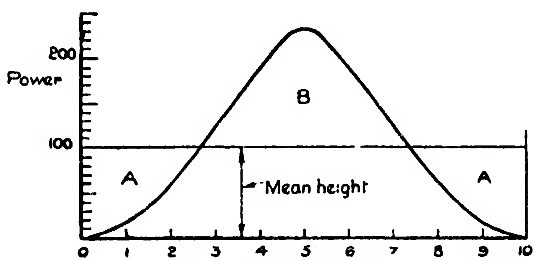


FIG. 39.

Let it next be required to find the mean value of a function – say, the mean value of y when $y = 4x^2+7x-5$, the range of x being 1 to 6. We have seen that it is really necessary first to find the area under the curve $y = 4x^2+7x-5$ within the proper boundaries and then to divide by the length of the base; and since the relation between y and x is stated, it is possible to dispense with the graph and work entirely by algebraic integration; thus also ensuring the true result, for in reality the mean is automatically taken of an infinite number of ordinates.

In the case taken as an illustration, the base is the axis of x, or, more strictly, the portion of it between $x = 1$ and $x = 6$, so that the length of the base $= 6-1 = 5$ units, and the area between the curve, the axis of x and the ordinates through $x = 1$ and $x = 6$ is given by the value of $\int_1^6 y\,dx$, *i.e.*, $\int_1^6 (4x^2+7x-5)\,dx$,

so that the mean value (for which we shall write m.v.) is—

$$\text{m.v.} = \frac{\int_1^6 (4x^2+7x-5)\,dx}{5}$$

$$= \frac{\left[\frac{4x^3}{3}+\frac{7x^2}{2}-5x\right]_1^6}{5}$$

$$= \frac{\left(\frac{4\times 216}{3}\right)+\left(\frac{7\times 36}{2}\right)-30-\frac{4}{3}-\frac{7}{2}+5}{5}$$

$$= \frac{384\cdot 17}{5} = 76\cdot 83.$$

It is instructive to compare this result with the results obtained by the use of the mid-ordinate rule :—

(*a*) Taking 5 ordinates only, we have the values—

x	$4x^2+7x-5$	y
$1\frac{1}{2}$	$9+10\cdot 5-5$	$14\cdot 5$
$2\frac{1}{2}$	$25+17\cdot 5-5$	$37\cdot 5$
$3\frac{1}{2}$	$49+24\cdot 5-5$	$68\cdot 5$
$4\frac{1}{2}$	$81+31\cdot 5-5$	$107\cdot 5$
$5\frac{1}{2}$	$121+38\cdot 5-5$	$154\cdot 5$

Their sum = $382\cdot 5$

and the average = $76\cdot 5$.

(*b*) Taking 10 ordinates, viz., those at $x = 1\frac{1}{4}$, $1\frac{3}{4}$, $2\frac{1}{4}$, etc., the values of y are 10, 19·5, 31, 44·5, 60, 77·5, 97, 118·5, 142 and 167·5

their sum = 767·5

and the average = 76·75.

Therefore, by increasing the number of ordinates measured, a better approximation is found.

The curve is a parabola with axis vertical, and hence Simpson's rule should give the result accurately if 3 ordinates only are taken, viz., at $x = 1$, 3·5 and 6.

Thus, A = 6, M = 68·5, B = 181.

$$\text{Hence the mean height} = \frac{6+(4\times 68\cdot 5)+181}{6}$$

$$= \frac{461}{6} \quad 76\cdot 83.$$

If, then, the law connecting the two variables is known, the mean value of the one over any range of the other can be found by integrating the former with regard to the latter between the proper limits and then dividing by the range; or to express in symbols, if $y = f(x)$, the mean value of y, as x ranges from a to b, is given by—

$$\frac{\int_a^b y\,dx}{b-a}$$

Example 2.—Find the mean value of e^{5x} between $x = \cdot 2$ and $x = \cdot 7$

$$\text{m.v.} = \frac{\int_{\cdot 2}^{\cdot 7} e^{5x}\,dx}{\cdot 7 - \cdot 2} = \frac{1}{\cdot 5}\left\{\frac{1}{5}e^{5x}\right\}_{\cdot 2}^{\cdot 7}$$
$$= \frac{2}{5}\{e^{3\cdot 5} - e^{1\cdot 0}\}$$
$$= \frac{2}{5}\{33\cdot 12 - 2\cdot 72\}$$
$$= \underline{\underline{12\cdot 16}}$$

i. e., if the curve $y = e^{5x}$ were plotted between $x = \cdot 2$ and $x = \cdot 7$ its mean ordinate would be 12·16 units.

Example 3.—If $C = 5 \sin 3t$, find the m.v. of C, when—

(*a*) t varies from 0 to $\frac{2\pi}{3}$

(*b*) t varies from 0 to $\frac{\pi}{3}$.

Whenever dealing with the integration of trigonometric functions it is advisable first to determine the period of the function, since much numerical work may often be saved in this way. The sine and cosine curves are curves symmetrical about the axis of the I.V. (*i. e.*, x or t, as the case may be); hence there is as much area above this axis as below it, if a full period or a multiple of periods be considered. Therefore, regarding the area above the x axis as positive and the area below this axis as negative, the net area over the full period is zero, so that the mean height of the curve, and therefore the m.v. of the function, must be zero.

For the case of $C = 5 \sin 3t$, the period $= \dfrac{2\pi}{\text{coeff}^{\text{t}}.\text{ of } t} = \dfrac{2\pi}{3}$

and hence if the m.v. is required for t ranging from 0 to $\frac{2\pi}{3}$

0 to $\frac{4\pi}{3}$, etc.

the result is zero.

Hence, whenever the analysis shows that the full period is involved, there is no need to go through the process of integration. In this case, however, the integration is performed for purposes of verification.

(a) $$\text{m.v. of C.} = \int_0^{\frac{2\pi}{3}} \frac{5 \sin 3t\,dt}{\frac{2\pi}{3} - 0} = \frac{3 \times 5}{2\pi}\left(-\frac{1}{3}\cos 3t\right)_0^{\frac{2\pi}{3}}$$

$$= -\frac{5}{2\pi}(\cos 2\pi - \cos 0)$$

$$= -\frac{5}{2\pi}(1 - 1)$$

$$= \underline{0.}$$

(b) $$\text{m.v. of C.} = \int_0^{\frac{\pi}{3}} \frac{5 \sin 3t\,dt}{\frac{\pi}{3} - 0} = \frac{3}{\pi} \times 5\left(-\frac{1}{3}\cos 3t\right)_0^{\frac{\pi}{3}}$$

$$= -\frac{5}{\pi}(\cos \pi - \cos 0)$$

$$= -\frac{5}{\pi}(-1 - 1) = \underline{\frac{10}{\pi}}.$$

Comparing this with the amplitude, which is the maximum ordinate of the curve and has the value $5 \sin \frac{\pi}{2} = 5$, we note that—

$$\frac{\text{Mean ordinate for } \frac{1}{2} \text{ period}}{\text{Maximum ordinate for } \frac{1}{2} \text{ period}} = \frac{10}{\pi \times 5} = \frac{2}{\pi} = \cdot 637.$$

The average height of a sine curve is always $\cdot 637 \times$ the maximum height.

Example 4.—If an alternating electric current is given by the relation—

$$I = \cdot 5 \sin 120\pi t + \cdot 06 \sin 600\pi t$$

find the mean value of I.

The graph is of great assistance in this evaluation; and consequently the curves $i_1 = \cdot 5 \sin 120\pi t$, $i_2 = \cdot 06 \sin 600\pi t$, and $I = i_1 + i_2 = \cdot 5 \sin 120\pi t + \cdot 06 \sin 600\pi t$ are plotted in Fig. 40. The period of $i_1 = \cdot 5 \sin 120\pi t$ is $\frac{2\pi}{120\pi}$ or $\frac{1}{60}$, and of $i_2 = \cdot 06 \sin 600\pi t$ is $\frac{2\pi}{600\pi}$ or $\frac{1}{300}$, so that the period of the compound curve must be $\frac{1}{60}$.

From the previous reasoning it is seen that the mean height of the curve $i_1 = \cdot 5 \sin 120\pi t$ must be $\cdot 637 \times \text{amplitude} = \cdot 637 \times \cdot 5 = \cdot 3185$; and the mean height of the curve $i_2 = \cdot 06 \sin 600\pi t$ considered over the same period, viz., 0 to $\frac{1}{120}$, must be the mean height of the wave A, since the positive and negative areas are otherwise balanced, but this

must be spread over five times its usual base; now the mean height of the wave A is $\cdot 637 \times \cdot 06$, so that the mean height of the curve $i_2 = \cdot 06 \sin 600\pi t$ over the period 0 to $\frac{1}{120}$ is $\frac{\cdot 637 \times \cdot 06}{5}$ or $\cdot 0076$.

Then the mean value of $I = \cdot 3185 + \cdot 0076 = \underline{\cdot 3261}$.

The nature of the average should be clearly understood; for it is possible that the same quantity may have two different averages according to the way those averages are considered. Suppose a piston is pushed by a variable force; then the average value of that force might be found by taking readings of it at every foot of the stroke and dividing by the number of readings taken, in

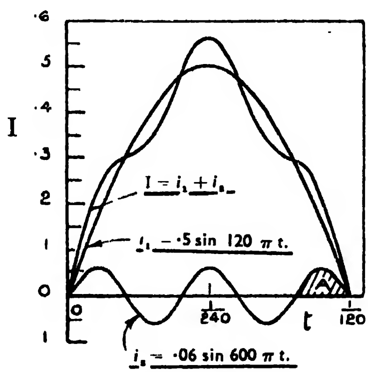


FIG. 40.—Problem on Alternating Current.

which case the average would be termed a space average; or the force might be measured at equal intervals of time, whence the time average of the force would result.

To take another instance:—Suppose a bullet penetrates a target to a depth of s ft.; the average value of the force calculated from the formula $Ps = \frac{wv^2}{2g}$, where P is the force, would be a space average, but P calculated from the formula $Pt = \frac{wv}{g}$, *i.e.*, the force causing change of momentum in a definite time, would be a time average.

To illustrate further: A body starts from rest and its speed increases at the uniform rate of 4 ft. per sec.². Find the time and space averages of the velocity if the motion is considered to take

place for 6 secs. {Use the relations $s = \frac{1}{2}at^2$, $v = at$ and $v^2 = 2as$.}

$$s = \frac{1}{2}at^2 = \frac{1}{2}\times 4\times 6^2 = 72$$

and if the limits of t are 0 and 6

then the limits of s are 0 and 72.

To find the time average of the velocity—

$$\text{t.a.} = \frac{\int_0^6 v dt}{6-0} = \frac{\int_0^6 at\,dt}{6}$$

$$= \frac{1}{6}\left\{\frac{1}{2}at^2\right\}_0^6 = \frac{1}{12}\times 4\{36-0\}$$

$$= 12 \text{ f.p.s.}$$

To find the space average of the velocity—

$$v^2 = 2as = 8s$$

or

$$v = \sqrt{8}\sqrt{s}$$

and the mean value of v = mean value of $\sqrt{8}s^{\frac{1}{2}}$.

$$\text{Hence the space average} = \frac{\int_0^{72}\sqrt{8}s^{\frac{1}{2}}ds}{72-0} = \frac{1}{72}\times\sqrt{8}\left\{\frac{2}{3}s^{\frac{3}{2}}\right\}_0^{72}$$

$$= \frac{\sqrt{8}}{72}\times\frac{2}{3}\{72^{\frac{3}{2}}-0\}$$

$$= \frac{\sqrt{8}}{108}\{432\sqrt{2}\}$$

$$= \frac{432\times 4}{108} = 16 \text{ f.p.s.}$$

Example 5.—The electrical resistance R_t of a rheostat at temperature $t°$ C. is given by—

$$R_t = 38(1+\cdot 004t).$$

Find its average resistance as t varies from 10° to 40° C. (This will be a temperature average.)

$$\text{m.v.} = \frac{\int_{10}^{40} R_t dt}{40-10} = \frac{1}{30}\int_{10}^{40} 38(1+\cdot 004t)dt$$

$$= \frac{38}{30}\left(t+\cdot 002t^2\right)_{10}^{40}$$

$$= \frac{38}{30}(40+3\cdot 2-10-\cdot 2)$$

$$= \frac{38\times 33}{30}$$

$$= \underline{41\cdot 8.}$$

Example 6.—If $V = V_0 \sin qt$ and $I = I_0 \sin(qt-\phi)$, find the average value of the power, *i. e.*, the average value of VI.

$$
\begin{aligned}
VI &= V_0 \sin qt \,.\, I_0 (\sin qt-\phi)\\
&= \frac{V_0 I_0}{2}\{2 \sin qt \,.\, \sin(qt-\phi)\}\\
&= \frac{V_0 I_0}{2}\{\cos\phi - \cos(2qt-\phi)\}
\end{aligned}
$$

also the period $= \dfrac{2\pi}{q}$.

Hence the m.v. of VI

$$
\begin{aligned}
&= \frac{\dfrac{V_0 I_0}{2}\displaystyle\int_0^{\frac{2\pi}{q}} [\cos\phi - \cos(2qt-\phi)]dt}{\dfrac{2\pi}{q}}\\
&= \frac{V_0 I_0 q}{2\times 2\pi}\left[\cos\phi \times t - \frac{1}{2q}\sin(2qt-\phi)\right]_0^{\frac{2\pi}{q}}\\
&= \frac{q}{2\pi}\times\frac{V_0 I_0}{2}\left\{\frac{2\pi}{q}\cos\phi - \frac{1}{2q}\sin(4\pi-\phi) - 0 + \frac{1}{2q}\sin(-\phi)\right\}\\
&= \frac{V_0 I_0 q}{4\pi}\times\frac{2\pi}{q}\cos\phi \qquad \left[\begin{array}{ll}\text{for} & \sin(4\pi-\phi) = \sin(-\phi)\\ \text{and} & -\sin(-\phi)+\sin(-\phi) = 0\end{array}\right]\\
&= \frac{1}{2} V_0 I_0 \cos\phi
\end{aligned}
$$

i.e., the mean value of the power = one-half the products of the maximum values or amplitudes with the cosine of the lag.

This is a most important result.

If $\phi = 90^\circ$, *i. e.*, if the lag is $\frac{\pi}{2}$, then $\cos\phi = 0$ and the mean value of the watts $= \frac{1}{2} V_0 I_0 \times 0 = 0$; this being spoken of as the case of *wattless current.*

Exercises 16.—On Mean Values.

1. If a gas expands so as to follow the law $pv = 120$, find the average pressure between the volumes 2 and 4.

2. Find the mean value of $e^{2 \cdot 5y}$ as y varies from 0 to ·4.

3. The mean height of the curve $y = 3x^3+5x-7$ is required between the limits $x = -2$ and $x = +3$; find this height.

4. Find the mean value of $2 \cdot 18 \sin(3t - 1 \cdot 6)$ as t ranges from ·14 to 1·6.

5. What is the mean value of $4 \cdot 5 + 2 \sin 60t$, t ranging from 0 to $\frac{\pi}{30}$? Discuss this question from its graphic aspect.

6. Find the mean value of p, when $pv^{1 \cdot 37} = 550$, for the range of v from 4 to 22.

7. The illumination I (foot candles) of a single arc placed 22 ft. above the ground, at d ft. from the foot of the lamp is given by $I = 1 \cdot 4 - \cdot 01 d$. Find the mean illumination as d varies from $\frac{1}{2}$ ft. to 10 ft.

8. An alternating current is given by $I = \cdot 2 \sin 100\pi t + \cdot 01 \sin 300\pi t$. Find the mean value of I for the range of t, 0 to $\cdot 02$ sec.

9. Taking the figures in Question 8, find the mean value of I when t ranges from 0 to $\cdot 01$ sec.

10. Find the mean value of $5 \sin 6t \times 220 \sin 4t$, t ranging from 0 to $\frac{\pi}{3}$.

11. The table gives the values of the side thrust on the piston of a 160 H.P. Mercédès aero engine for different positions of the crank; the positive values being the thrust on the right-hand wall, and the negative values being the thrust on the left-hand wall of the piston.

Angle of crank from top dead centre	0	40	80	100	120	140	160	180	200	240	280	300	320	330
Total side thrust (lbs.)	0	+210	0	−170	−175	−185	−100	0	+95	+240	+100	0	−30	0

345	360	10	40	80	120	160	180	200	220	240	260	280	290	320	360
+20	0	−600	−780	−720	−580	−170	0	+90	+170	+160	+150	+50	0	−210	0

Plot these values (treating them all as positive) and thence determine the mean side thrust throughout the cycle.

Root Mean Square Values.—A direct electric current may be measured by its three effects—chemical decomposition, magnetic effect or heating effect. An alternating current (A.C.), however, flowing first in one direction and then in the reverse, cannot be measured by either the first or the second of these effects, because the effect due to the flow in one direction would be neutralised by that due to the opposite flow ; hence an A.C. must be measured by its heating effect.

The heating effect of a current expressed by the heat units H may be measured by—

$$H = \frac{I^2 R t}{J} \text{ where I is the current,}$$

so that it will be seen that $H \propto I^2$.

The measuring instruments are graduated to give the root of the mean value of this heating effect, *i. e.*, the square root of the

mean value of the squares (called the root mean square value and written R.M.S.); in other words, the instrument records what are termed virtual amperes, a virtual ampere being the current that produces the same heat in a resistance as a steady or direct current of 1 ampere in the same time.

In place of the measurement of the current by its heating effect, the ammeter might be of the electro-dynamometer type, and in such the instrument records the mean value of I^2, and the square root of this value is called the effective current.

Similar remarks apply also to the measurement of alternating E.M.F.

It is therefore necessary to determine R.M.S. values of functions likely to be encountered, and to compare the virtual with the steady.

While the determination of R.M.S. values is of greatest importance from the application to electrical problems, it is also occasionally of use in problems of mechanics; thus the calculation of what is known as a swing radius (see p. 240) is in reality a determination of a R.M.S. value.

In order to convey the full interpretation of the term R.M.S. value, we shall discuss first a simple arithmetical example, then some algebraic examples, leading up finally to the trigonometric functions.

Example 7.—The values of an alternating electric current at various times are given in the table—

t	0	·01	·02	·03	·04	·05	·06	·07	·08	·09	·1
C	0	1·8	3·5	5	6·1	6·8	6·1	5	3·5	1·8	0

Find the mean value and also the R.M.S. value of the current, and compare the two values.

The mean value, as before explained, is the average of the given values and is 3·96.

We must now tabulate the values of the squares; thus—

0 3·24 12·25 25 37·21 46·24 37·21 25 12·25 3·24 and 0.

The sum of the squares $= 201·64$

$$\text{the mean of the squares or M.S.} = \frac{201·64}{10}$$

for we must only reckon the end values as half-values when adding

up the ordinates, since the end values belong equally to the sequences on either side.

Thus— M.S. $= 20{\cdot}16$

and the square root of the result $=$ R.M.S. $= \sqrt{20{\cdot}16} = \underline{4{\cdot}49}$

This question might have been worked entirely by graphic methods, according to the following plan—

Plot the values of C to a base of t, giving the curve ABD in Fig. 41; find the area under the curve and divide by the base, thus obtaining the mean height; plot also the values of C^2 against those of t, giving the curve EFG. By graphic summation determine the area under the curve EFG and draw the line MM at the mean height of the diagram. Make MN $=$ 1 unit on the scale of C^2 and on PN describe a semicircle; produce MM to cut the semicircle in R. Then MR $= 4{\cdot}5$ is the R.M.S. value.

Thus the mean value $= 3{\cdot}96$ and the R.M.S. value $= 4{\cdot}5$, and the ratio $\dfrac{\text{R.M.S. value}}{\text{mean value}} = \dfrac{4{\cdot}5}{3{\cdot}96} = \underline{1{\cdot}14}$; this ratio being termed the *form factor*.

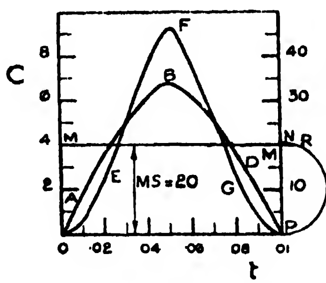


FIG. 41.—R.M.S. Value of an Alternating Current.

Example 8.—Find the R.M.S. value of the function $2x^{1\cdot5}-3x$ as x ranges from 2 to 5.

The square $= (2x^{1\cdot5}-3x)^2 = 4x^3+9x^2-12x^{2\cdot5}$

and the M.S. $= \dfrac{\int_2^5 (4x^3+9x^2-12x^{2\cdot5})\,dx}{5-2}$

$$= \frac{1}{3}\left(x^4+3x^3-3{\cdot}43x^{3\cdot5}\right)_2^5$$

$$= \frac{1}{3}\left[(625+375-959)-(16+24-38{\cdot}8)\right]$$

$= 13{\cdot}27$.

Hence the R.M.S. value—

$= \sqrt{13{\cdot}27} = \underline{3{\cdot}64}$.

Explanation.

No.	
5	·6990
	3·5
	34950
	20970
	2·44650
3·43	·5353
959	2·9818
2	·3010
	3·5
	15050
	9030
	1·05350
3·43	·5353
38·8	1·5888

Example 9.—Suppose that an alternating electric current at any time follows the sine law, *i. e.*—

$$I = I_0 \sin \omega t$$

where I is the instantaneous value of the current at any time t, and I_0 is the maximum value of the current.

Find the R.M.S. value of the current.

As we have already seen, the determination of the R.M.S. value implies that first the square of the function at various times must be calculated, then the mean value of these squares found, and finally the square root of this average extracted.

To assist in the study of this important problem, the curve $y = \sin x$, the simple sine curve, is shown in Fig. 42, and also the

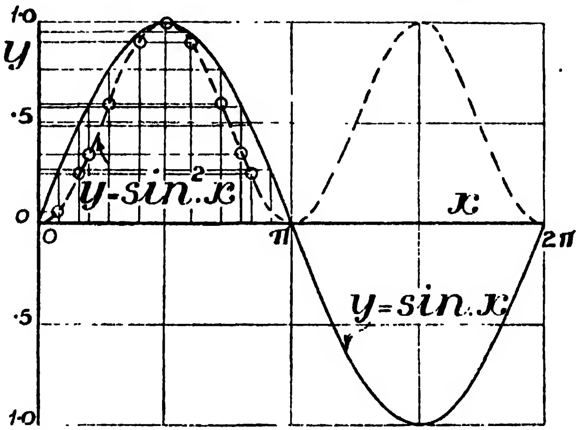


FIG. 42.—R.M.S. Value of an Alternating Current.

curve $y = \sin^2 x$, which is obtained from the former curve by squaring its ordinates. It will be observed that whilst for the curve $y = \sin x$ there are both positive and negative ordinates, in the case of the curve of squares all the ordinates are positive. Also the period of the curve of squares is noted to be one-half that of the simple curve; and therefore when calculating the mean height of this curve, it is immaterial whether the full or the half period of the simple sine curve is taken as the range.

In the case with which we are here particularly concerned the square $= I^2 = I_0^2 \sin^2 \omega t = \frac{I_0^2}{2}(1 - \cos 2\omega t)$, and the period $= \frac{2\pi}{\omega}$, so that the integration may be performed either over the range 0 to $\frac{2\pi}{\omega}$ or 0 to $\frac{\pi}{\omega}$.

Taking the latter range—

$$\text{Mean of the squares} = \text{M.S.} = \frac{\int_0^{\frac{\pi}{\omega}} I_0^2 \sin^2 \omega t\, dt}{\frac{\pi}{\omega}}$$

$$= \frac{\omega I_0^2}{\pi}\int_0^{\frac{\pi}{\omega}} (1 - \cos 2\omega t)\, dt \text{ *}$$

$$= \frac{\omega I_0^2}{2\pi}\left[\left(t - \frac{1}{2\omega}\sin 2\omega t\right)\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{\omega I_0^2}{2\pi}\left[\frac{\pi}{\omega} - \frac{1}{2\omega}\sin 2\pi - 0 + \frac{1}{2\omega}\sin 0\right]$$

$$= \frac{\omega I_0^2}{2\pi} \times \frac{\pi}{\omega} = \frac{I_0^2}{2}.$$

$$\text{R.M.S.} = \frac{I_0}{\sqrt{2}} = \underline{\cdot 707\ I_0.}$$

or R.M.S. value $= \cdot 707 \times$ maximum value

i.e., virtual value of current or E.M.F. $= \cdot 707 \times$ maximum value of current or E.M.F.

Hence if a meter registers 10 amp., the maximum current is $\frac{10}{\cdot 707}$, *i. e.*, 14·14 amp., or there is a variation between $+14\cdot 14$ and $-14\cdot 14$.

---

* In the evaluation of the integral $\int_0^{\frac{\pi}{\omega}}(1 - \cos 2\omega t)\, dt$, it should be noted that it can be written $\int_0^{\frac{\pi}{\omega}} dt - \int_0^{\frac{\pi}{\omega}} \cos 2\omega t\, dt$, and the value of the second term is zero, because it is the area under a cosine curve taken over its full period. Hence the integral reduces to $\int_0^{\frac{\pi}{\omega}} dt$, and there is no need to say anything further about the second term.

***Example* 10.—Find the R.M.S. value of $a + b \sin 4t$.**

For this function the period $= \frac{2\pi}{4} = \frac{\pi}{2}$.

Then $S = (a + b \sin 4t)^2 = a^2 + b^2 \sin^2 4t + 2ab \sin 4t$

$$= a^2 + \frac{b^2}{2}(1 - \cos 8t) + 2ab \sin 4t.$$

Hence $\text{M.S.} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(a^2 + \frac{b^2}{2} - \frac{b^2}{2}\cos 8t + 2ab \sin 4t\right) dt$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\left(a^2 + \frac{b^2}{2}\right) dt - \int_0^{\frac{\pi}{2}} \frac{b^2}{2}\cos 8t dt + 2ab\int_0^{\frac{\pi}{2}} \sin 4t\, dt\right]$$

$$= \frac{2}{\pi}\left[a^2t+\frac{b^2t}{2}-0+0\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left(a^2\frac{\pi}{2}+\frac{b^2}{2}\frac{\pi}{2}-0-0\right) = a^2+\frac{b^2}{2}$$

$$\therefore \quad \text{R.M.S.} = \underline{\sqrt{a^2+\frac{b^2}{2}}}.$$

Exercises 17.—On Root Mean Square Values.

Find the R.M.S. values of the functions in Nos. 1 to 7.

1. x^3+2 (x ranging from 1 to 3).
2. e^{-7x} (x ranging from $-\cdot 1$ to $+\cdot 65$).
3. $3\cdot 4 \sin 5\cdot 1t$. 4. $\cdot 165 \cos(\cdot 07-2t)$.
5. $1\cdot 4 \tan 2t$ (t ranging from 0 to $\cdot 43$).
6. $1\cdot 14+\cdot 5 \cos \cdot 8t$.
7. $\cdot 72 \sin(3-4t)$; compare with the mean value.
8. Find the form factor $\left(\frac{\text{R.M.S. value}}{\text{mean value}}\right)$ of the wave—

$$e = E_1 \sin \omega t+E_2 \sin 3\omega t.$$

9. Compare the "effective" values of two currents, one whose wave form is sinusoidal, having a maximum of 100 amperes, and another of triangular shape with a maximum of 150 amperes.

10. An A.C. has the following values at equal intervals of time: 3, 4, 4·5, 5·5, 8, 10, 6, 0, −3, −4, −4·5, −5·5, −8, −10, −6, 0. Find the R.M.S. value of this current.

11. A number of equal masses are attached to the ends of rods rotating about one axis. If the lengths of these rods are 10, 9, 5, 8, 4, 13 and 15 ins. respectively, find the effective radius (called the swing radius) of the system. (This is the R.M.S. value of the respective radii.)

12. Find the R.M.S. value of the function $\sin^2\theta\cos^3\theta$ over the period 0 to $\frac{\pi}{2}$. (Refer to p. 178.)

13. The value of the primary current through a transformer at equal time intervals was—

·20 ·05 ·07 ·11 ·14 ·19 ·21 ·04 ·08 12 ·15 ·18 ·21 ·08 ·04

Find the R.M.S. value of this current.

Volumes.—If a curve be drawn, the ordinates being the values of the cross sections of a solid at the various points along its length, then the area under the curve will represent, to some scale, the volume of the solid. For, considering a small element,

the volume δV of the small portion of the solid is $A\delta l$, and the total volume of the solid is the sum of all such small volumes, *i. e.*, $\Sigma A\delta l$. If the length δl is diminished until infinitely small, $\Sigma A\delta l$ becomes $\int A dl$, and hence—

$$\text{Volume of solid} = \int_0^l A dl.$$

Comparing this with the formula giving the area of a closed figure, viz., $\int y dx$, it is seen to be of the same form, and it is made identical if A is written in place of y (*i. e.*, areas must be plotted vertically) and l in place of x (*i. e.*, lengths must be plotted horizontally).

When values of A and l are given the curve should be drawn and integrated by any of the given methods; and if it is preferred to find the actual area of the figure in sq. ins. first, the number of units of volume represented will be deduced from a consideration of the scales used in the plotting. Thus if 1″ (horizontally) represents x ft. of length, and 1″ (vertically) represents y sq. ft. of area, then 1 sq. in. of area represents xy cu. ft. of volume. To take a numerical example: If 1″ = 15″ of length, and 1″ = 5 sq. ft. of area, then 1 sq. in. of area $= \frac{15}{12}\times 5 = 6\frac{1}{4}$ cu. ft. of volume.

If the area is found by the sum curve this conversion is unnecessary, as the scales are settled in the course of the drawing. The Example on p. 122 is an illustration of the determination of volumes by graphic integration; in that case the actual areas of sections are not plotted directly, but values of d^2, the multiplication by the constant factor being left until the end. Had the solid not been of circular section the actual areas would have been plotted as ordinates and the work carried on as there detailed.

We thus see that the determination of the volume of any irregular solid can be effected, if the cross sections at various distances from the ends can be found, by a process of graphic integration.

If, however, the law governing the variation in section is known, it may be more direct to perform the integration by algebraic methods.

To take a very simple illustration :—

Example 11.—The cross section of a certain body is always equal to $(5x^2+8)$ sq. ft., where x ft. is the distance of the section from one end. If the length of the body is 5 ft., find its volume.

The body might have an elevation like Fig. 43, and its cross section might be of any shape; the only condition to be satisfied being that the area of a cross section such as that at BB must $= 5x^3+8$. Thus the area at AA must $= (5\times 0)+8 = 8$ and area at CC $= (5\times 5^3)+8 = 633$.

$$\text{Then the volume} = \int_0^5 A\,dx = \int_0^5 (5x^3+8)\,dx$$

$$= \left(\frac{5x^4}{4}+8x\right)_0^5 = \frac{3125}{4}+40 = \underline{\underline{821\cdot25 \text{ cu. ft.}}}$$

Volumes of Solids of Revolution.—A solid of revolution is generated by the revolution of some closed figure round an axis which does not cut the figure. Thus, dealing with familiar solids, the right circular cone, the cylinder and the sphere are solids of revolution, being generated respectively by the revolution of a right-angled triangle about one of the sides including the right angle, a rectangle about one of its sides, and a semicircle about its diameter; and of the less well-known solids of revolution the most important to the engineer is the hyperboloid of revolution which is generated by the revolution of a hyperbola about one of its axes and occurs in the design of skew wheels. The axis about which the revolution is made in all these examples lies along a boundary of the revolving figure; whereas an anchor ring is generated by the revolution of a circle about an axis parallel to a diameter but some distance from it.

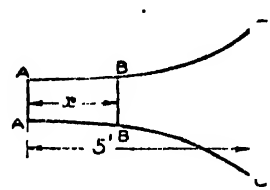


FIG. 43.

The revolving figure may have any shape whatever, the only conditions being, for the following rule to hold, that the axis about which the revolution is made does not cut the figure and that the cross section perpendicular to this axis is always circular.

Imagine the revolving cross section to be of the character shown in the sketch (ABCD in Fig. 44); and let the revolution be about the axis of x. It is required to find the volume of the solid of revolution generated.

Working entirely from first principles, *i. e.*, reverting to our idea of dealing with a small element and then summing; if the strip MN of height y and thickness δx revolves about OX it will generate a cylinder.

The radius of this cylinder will be y and its height or length δx; and hence its volume $= \pi y^2 \delta x$.

Accordingly the volume swept out by the revolution of ABCD

will be $\Sigma \pi y^2 \delta x$ (the proper limits being assigned to x) approximately, or $\int \pi y^2 dx$ accurately.

Again, it will be seen that such a volume can be measured by the area of a figure, for, writing Y in place of πy^2, the volume $= \int Y dx$, which is the standard form for an area. Hence, if values of y are given, corresponding values of πy^2 must be calculated and plotted as ordinates, and the area of the resulting figure found.

It should be noted that $\int \pi y^2 dx$ might be written as $\pi \int y^2 dx$, thus saving labour by reserving the multiplication by π until the

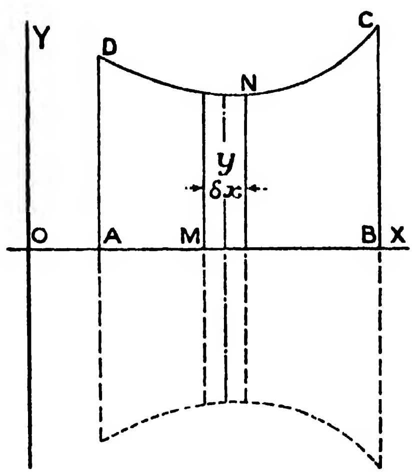


FIG. 44.—Volume of Solid of Revolution.

area has been found, *i. e.*, the values of y^2 and not those of πy^2 are plotted as ordinates.

The following example will illustrate :—

Example 12.—The curve given by the figures in the table revolves about the axis of x; find the volume of the solid generated, the bounding planes being those through $x = 2$ and $x = 7$, perpendicular to the axis of revolution.

x	2	3	4	5	6	7
y	44	42	44	46	45	38

Values of y^2 must first be calculated, since the volume $= \int_2^7 \pi y^2 dx$

$$= \pi \int_2^7 y^2 dx$$

Hence the table for plotting reads—

x	2	3	4	5	6	7
y^2	1936	1764	1936	2116	2025	1444

and the values of y^2 are plotted vertically, the curve ABC (Fig. 45) resulting.

This curve is next integrated from the axis of x as base, the curve DEF resulting; the polar distance being taken as 3, so that the new

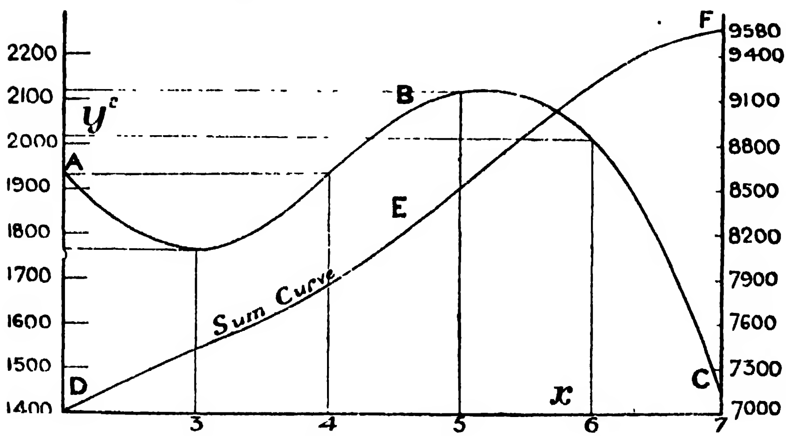


FIG. 45.—Volume of Solid of Revolution.

vertical scale $= 3 \times$ old vertical scale. It must be remembered, however, that the base from which the summation has been made in the figure is not the true base, since the first value of the ordinate is 1400 and not 0; thus a rectangle 1400×5 has been omitted. Hence we must start to number our scale at 7000; and according to this numbering the last ordinate reads 9580, hence $\int_2^7 y^2 dx = 9580$, or—

$$\begin{aligned} \text{Volume} &= \pi \int_2^7 y^2 dx \\ &= \pi \times 9580 \\ &= \underline{30{,}100 \text{ cu. ins.}} \end{aligned}$$

In cases in which y and x are connected by a law the integra-

tion may be performed in accordance with the rules for the integration of functions.

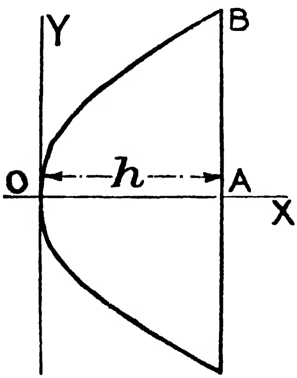


FIG. 46.

Example 13.—Find the volume of a paraboloid of revolution and compare it with the volume of the circumscribing cylinder.

A paraboloid of revolution is generated by the revolution of a parabola about its axis. Suppose the parabola is placed as shown in Fig. 46; the revolution is therefore about OX.

The equation to the curve OB is $y^2 = 4ax$ (see Part I, p. 106), *i. e.*, if $OA = h$, $(AB)^2 = 4ah$.

Hence the volume of the solid swept out by the revolution of OBA—

$$= \int_0^h \pi y^2 dx = \pi \int_0^h 4ax\, dx$$

$$= 4a\pi\left(\frac{x^2}{2}\right)^h$$

$$= \frac{4a\pi \times h^2}{2} = \underline{2a\pi h^2.}$$

Now the volume of the circumscribing cylinder—

$$= \pi(AB)^2 \times h$$

$$= \pi \times 4ah \times h$$

$$= 4a\pi h^2$$

and hence the volume of the paraboloid—

$$= \underline{\frac{1}{2} \times \text{vol. of circumscribing cylinder.}}$$

Example 14.—The curve $y^2 = 64 - 2x^2$ revolves about the axis of y. Find the volume of the solid generated, the limits to be applied to x being 0 and 5.

This differs from the cases previously treated in that the revolution is to be about the y axis and not about the x axis.

Hence the volume $= \int \pi x^2 dy$ and not $\int \pi y^2 dx$, the y replacing x and *vice versa*. Also another point must be noted: the limits given are those for x, whereas the limits in the integral $\int \pi x^2 dy$ must apply to the I.V., which is now y. Therefore a preliminary calculation must be made to determine the corresponding limits of y—

$$y^2 = 64 - 2x^2$$

When $x = 0$, $y^2 = 64$, $y = \pm 8$

and when $x = 5$, $y^2 = 14$, $y = \pm 3{\cdot}74$.

The double signs occurring here may possibly confuse, but actually the equation given is that of an ellipse, symmetrical about the axes of x and y, and the volume required is the volume generated by the revolution of the two shaded portions (Fig. 47), which will be twice that generated by one of these; hence, taking the upper shaded portion, we use the positive limits, viz., 3·74 and 8.

$$\text{Then the volume} = \int_{3\cdot74}^{8} \pi x^2 dy = \pi\int_{3\cdot74}^{8}\left(32-\frac{y^2}{2}\right)dy$$
$$= \pi\left(32y-\frac{y^3}{6}\right)_{3\cdot74}^{8}$$
$$= \pi(256-85\cdot3-119\cdot5+8\cdot7)$$
$$= 59\cdot9\pi.$$

The solid due to the revolution of the lower portion will be also $59\cdot9\pi$, and hence the total volume generated $= 119\cdot8\pi =$ 376 cu. units.

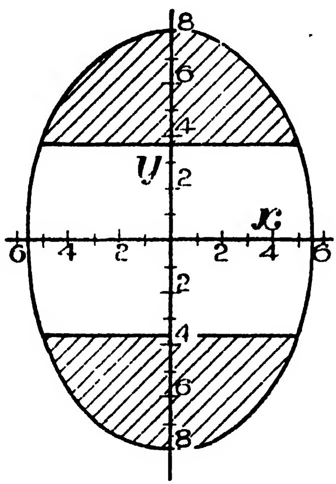


FIG. 47.

If the limits for y were -8 and $+8$, the volume of the whole solid would be required; then—

$$\text{Volume} = \int_{-8}^{8}\pi x^2 dy = 2\pi\int_{0}^{8}\left(32-\frac{y^2}{2}\right)dy = 2\pi\left[32y-\frac{y^3}{6}\right]_0^8$$
$$= 2\pi(256-85\cdot3)$$
$$= 341\cdot4\pi = \underline{1070 \text{ cu. units.}}$$

The solid generated by the revolution of an ellipse about its major axis is known as a *prolate spheroid*; while if the revolution is about the minor axis the solid is an *oblate spheroid*.

The volumes of these may be necessary, so that they are given in a general form.

The general equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (cf. Part I, p. 344).

Let $a>b$, so that the major axis is horizontal.

For a prolate spheroid the revolution is about the major axis, and the volume $= 2\int_0^a \pi y^2 dx \quad = 2\pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)dx$

$$= \frac{2\pi b^2}{a^2}\left(a^2x-\frac{x^3}{3}\right)_0^a$$

$$= \frac{2\pi b^2}{a^2}\times\frac{2}{3}a^3$$

$$= \frac{4}{3}\pi ab^2.$$

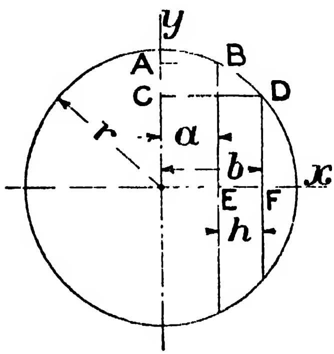


FIG. 48.

In like manner, the volume of an oblate spheroid $= \frac{4}{3}\pi a^2 b$; and it should be noted that if $b=a$, the spheroid becomes a sphere and its volume $= \frac{4}{3}\pi a^3$.

Example 15.—Find the volume of a zone of a sphere of radius r, the bounding planes being those through $x=a$ and $x=b$.

The equation of the circle is $x^2+y^2=r^2$ (Fig. 48)
whence $\qquad y^2 = r^2-x^2$.

$$\therefore \quad \text{Volume of a zone} = \int_a^b \pi y^2 dx = \pi\int_a^b (r^2-x^2)dx$$

$$= \pi\left(r^2x-\frac{x^3}{3}\right)_a^b$$

$$= \pi\left[r^2(b-a)-\frac{1}{3}(b^3-a^3)\right]$$

$$= \frac{\pi(b-a)}{3}[3r^2-(b^2+ab+a^2)].$$

This can be put in the form given on p. 120, Part I, if for $(b-a)$ we write h, and for BE and DF their respective values r_1 and r_2.

Thus $\qquad a^2 = r^2-r_1^2,\ b^2 = r^2-r_2^2$

$$h^2 = (b-a)^2 = b^2+a^2-2ab = r^2-r_1^2+r^2-r_2^2-2ab$$

and hence $\qquad ab = \dfrac{-h^2+2r^2-r_1^2-r_2^2}{2}$

So that the volume of the zone—

$$= \frac{\pi h}{3}\left\{3r^2-r^2+r_2^2-r^2+r_1^2+\frac{h^2-2r^2+r_1^2+r_2^2}{2}\right\}$$

$$= \frac{\pi h}{3}\left\{\frac{3}{2}r_2^2+\frac{3}{2}r_1^2+\frac{h^2}{2}\right\}$$

$$= \frac{\pi h}{6}\{3(r_1^2+r_2^2)+h^2\}.$$

Length of Arc.—Consider a small portion of a curve, PQ in Fig. 49, P and Q being points near to one another.

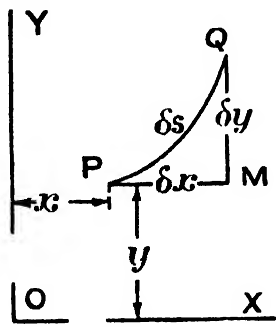


FIG. 49

The small length of arc PQ is denoted by δs, so that a complete arc would be denoted by s.

Let PM $= \delta x$, and QM $= \delta y$.

Then the arc PQ $=$ the chord PQ very nearly, so that we may say—

$$(\delta s)^2 = (\text{chord PQ})^2 = (\delta x)^2+(\delta y)^2.$$

$$\therefore \quad \left(\frac{\delta s}{\delta x}\right)^2 = 1+\left(\frac{\delta y}{\delta x}\right)^2 \quad \text{or} \quad \left(\frac{\delta s}{\delta y}\right)^2 = \left(\frac{\delta x}{\delta y}\right)^2+1$$

$$\textit{i.e.,} \quad \frac{\delta s}{\delta x} = \sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2} \quad \text{or} \quad \frac{\delta s}{\delta y} = \sqrt{1+\left(\frac{\delta x}{\delta y}\right)^2}.$$

Hence when $\delta x \to o$, $\dfrac{\delta s}{\delta x} \to \dfrac{ds}{dx}$, etc.,

$$\text{and} \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \text{or} \quad \frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}.$$

Integrating—

$$s = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, . \, dx \quad \text{or} \quad \int \sqrt{1+\left(\frac{dx}{dy}\right)^2} \, . \, dy.$$

The length of arc can thus be found if the value of the integral on the R.H.S. can be evaluated.

In only a few cases is the evaluation of the integral simple; and for most curves an approximation is taken, *e. g.*, to find the perimeter of an ellipse by this method one would become involved in a most difficult integral known as an elliptic integral, this being treated later in the chapter; and hence the approximate rules are nearly always used in practice.

To deal with a case of a very simple character :—

Example 16.—If $y = ax+b$, find the length of arc between $x = m$ and $x = n$.

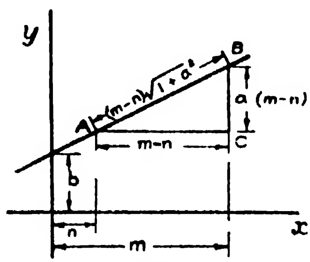


FIG. 50.

In this case it is really a matter of determining the length of the line AB (Fig. 50); the slope of the line being a.

Then if $y = ax+b$, $\frac{dy}{dx} = a$ and $1+\left(\frac{dy}{dx}\right)^2 = 1+a^2$.

Hence
$$= \int_n^m \sqrt{1+a^2}\, dx = \sqrt{1+a^2}\left(x\right)_n^m$$
$$= \underline{\sqrt{1+a^2}\ (m-n).}$$

On reference to the figure it will be seen that this is a true result,

since $\quad AB = \sqrt{(AC)^2+(CB)^2} = \sqrt{(m-n)^2+a^2(m-n)^2}$
$$= \sqrt{1+a^2}\ (m-n).$$

Example 17.—Use this method to determine the approximate length of a cable hanging in a parabola, when the droop is D and the span is 2L.

For convenience, put the figure in the form of Fig. 51.

Then $\quad L^2 = 4aD$

whence $\quad a = \frac{L^2}{4D}$, so that a must be very large.

The equation of the curve is in reality—

$$y^2 = 4ax$$

y being written in place of L, and x in place of D.

Then $$\frac{dy^2}{dx} = 4a, \text{ and also } \frac{dy^2}{dx} = \frac{dy^2}{dy} \times \frac{dy}{dx}$$

so that $$\frac{dy}{dx} \cdot 2y = 4a$$

or $$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

whence $$\frac{dx}{dy} = \frac{y}{2a}$$

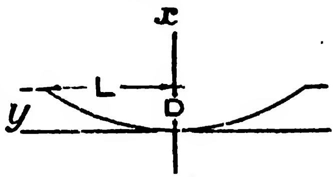


FIG. 51.

Thus

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{y^2}{4a^2}} = \left(1+\frac{y^2}{4a^2}\right)^{\frac{1}{2}}$$

$$= 1+\tfrac{1}{2} \cdot \frac{y^2}{4a^2}+\frac{\frac{1}{2}(-\frac{1}{2})}{1 \cdot 2}\left(\frac{y^2}{4a^2}\right)^2+\ldots$$

$$= 1+\frac{y^2}{8a^2} \text{ approximately,}$$

since all the subsequent terms contain a^4 and higher powers of a in the denominator, so that all these terms must be very small.

Hence $$s = 2\int_0^L \sqrt{1+\left(\frac{dx}{dy}\right)^2} \cdot dy$$

$$= 2\int_0^L \left(1+\frac{y^2}{8a^2}\right) dy$$

$$= 2\left(y+\frac{y^3}{24a^2}\right)_0^L$$

$$= 2\left(L+\frac{L^3}{24a^2}\right)$$

$$= 2\left(L+\frac{L^3 \times 16D^2}{24 \times L^4}\right) \qquad \text{for } a = \frac{L^2}{4D}$$

$$= \underline{2\,L+\frac{4D^2}{3L} \quad \text{or Span}+\frac{8\,(\text{Droop})^2}{3\text{ Span}}}.$$

Example 18.—Find the length of the circumference of a circle of radius r.

The equation of the circle is $y^2+x^2 = r^2$.

Thus $$y^2 = r^2-x^2$$

and $$2y \cdot \frac{dy}{dx} = -2x \quad \{\text{differentiating with regard to } x\}$$

or $$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{r^2-x^2}}$$

Hence $$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{x^2}{r^2-x^2} = \frac{r^2-x^2+x^2}{r^2-x^2}$$
$$= \frac{r^2}{r^2-x^2}$$

Length of circumference $= 4 \times$ length of $\frac{1}{4}$ circumference
$$= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.\,dx$$
$$= 4\int_0^a \frac{r}{\sqrt{r^2-x^2}}\,dx.$$

To evaluate this integral, let $x = r \sin u$ (cf. *Example* 7, p. 150).

Then $$r^2-x^2 = r^2-r^2\sin^2 u = r^2(1-\sin^2 u)$$
$$= r^2\cos^2 u$$

and also $$\frac{dx}{du} = r\cos u$$

i. e., $$dx = r\cos u\,.\,du.$$

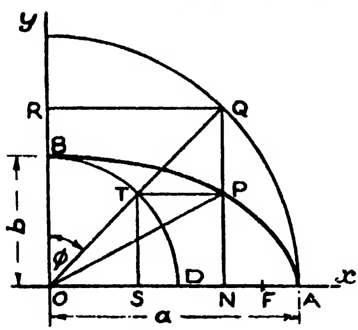


FIG. 52.

To find the limits to be assigned to u—

$$\sin u = \frac{x}{r};\ \text{and when } x = 0,\ u = 0$$
$$\text{and when } x = 1,\ u = 90^\circ \text{ or } \frac{\pi}{2}.$$

Thus the circumference $$= 4r\int_0^a \frac{dx}{\sqrt{r^2-x^2}}$$
$$= 4r\int_0^{\frac{\pi}{2}} \frac{r\cos u\,du}{r\cos u}$$
$$= 4r\Big(u\Big)_0^{\frac{\pi}{2}}$$
$$= \underline{2\pi r.}$$

Example 19.—Find an expression for the length of the perimeter of the ellipse whose major and minor axes are $2a$ and $2b$ respectively

Let BPA (Fig. 52) be the ellipse, CQA the quadrant of a circle on the major axis as diameter, and BTD the quadrant of a circle on the minor axis as diameter. Selecting P as any point on the ellipse, draw the lines QPN, PT, TS, QR and OTQ as shown.

The point P has the co-ordinates x and y, viz., ON and PN, which are respectively equal to QR and TS.

$$\begin{aligned} \text{Now} \quad & \text{QR} = \text{OQ} \sin\phi \quad \text{or} \quad x = a\sin\phi \\ \text{and} \quad & \text{TS} = \text{OT}\cos\phi \quad \text{or} \quad y = b\cos\phi. \end{aligned}$$

Thus $$\frac{dx}{d\phi} = a\cos\phi \quad \text{and} \quad \frac{dy}{d\phi} = -b\sin\phi$$

and $$\frac{ds}{d\phi} = \sqrt{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2} = \sqrt{a^2\cos^2\phi + b^2\sin^2\phi}$$

or $$s = \int \sqrt{a^2\cos^2\phi + b^2\sin^2\phi}\, d\phi.$$

Now the equation to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the eccentricity, which we shall denote by K, is given by—

$$\text{K} = \frac{\text{distance between foci}}{\text{major axis}} = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\text{OF}}{\text{OA}}, \quad \text{F being a focus.}$$

Hence $\text{K}^2 = \frac{a^2 - b^2}{a^2}$ and $\text{K}^2a^2 - a^2 = -b^2$

$$\text{or} \quad b^2 = a^2(1 - \text{K}^2)$$

so that $$\begin{aligned} a^2\cos^2\phi + b^2\sin^2\phi &= a^2\cos^2\phi + a^2\sin^2\phi - a^2\text{K}^2\sin^2\phi \\ &= a^2(1 - \text{K}^2\sin^2\phi). \end{aligned}$$

Thus our integral reduces to the form—

$$s = \int a\sqrt{1 - \text{K}^2\sin^2\phi}\, d\phi$$

and for the quarter of the ellipse the perimeter $= \int_0^{\frac{\pi}{2}} a\sqrt{1 - \text{K}^2\sin^2\phi}\, d\phi$, since the limits for ϕ are obviously 0 and $\frac{\pi}{2}$.

Also the full perimeter of the ellipse $= 4\int_0^{\frac{\pi}{2}} a\sqrt{1 - \text{K}^2\sin^2\phi}\, d\phi$.

This integral, called an "elliptic integral of the second kind," is extremely difficult to evaluate; but in view of the importance of the perimeter of the ellipse it is well that we should carry the work a little further.

Knowing the values of a and K for any particular ellipse, recourse may be made to tables of values of elliptic integrals, but if these are not available, a graphic method presents itself which is not at all difficult to use. According to this plan, various values of ϕ are chosen between 0 and $\frac{\pi}{2}$, and the calculated values of $\sqrt{1 - \text{K}^2\sin^2\phi}$ are plotted as ordinates to a base of ϕ. Then the area under the resulting curve when multiplied by $4a$ gives the perimeter of the ellipse.

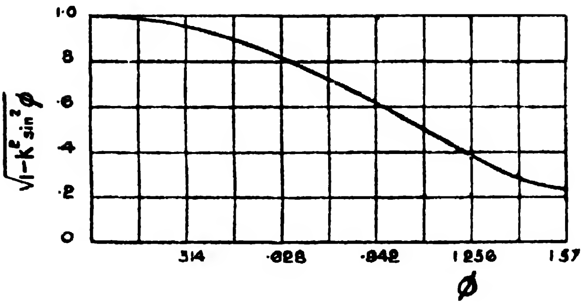


FIG 53.—Perimeter of Ellipse.

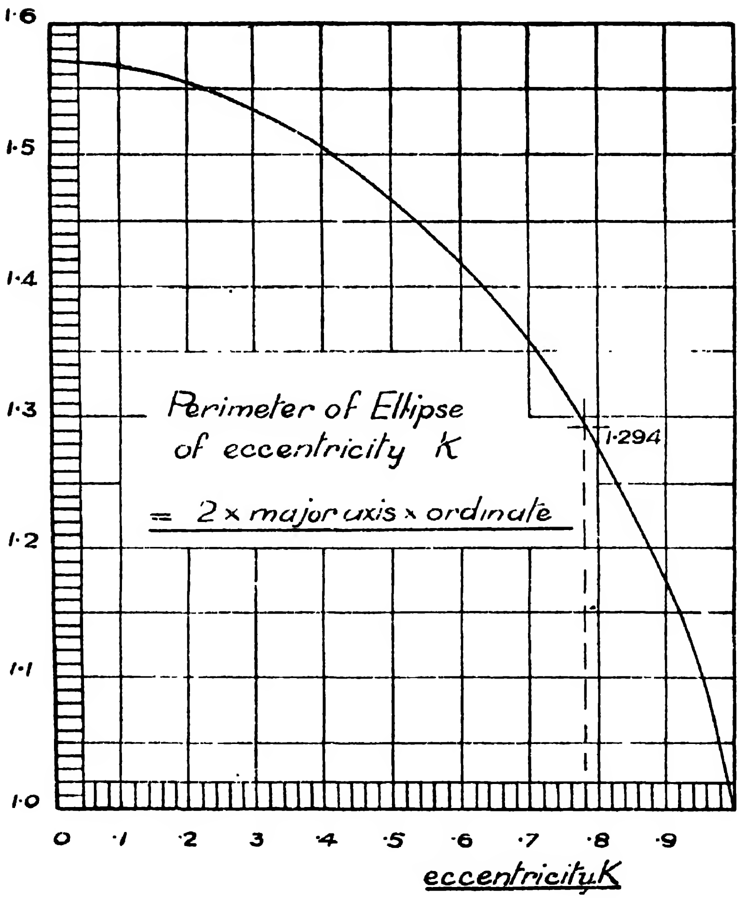


FIG. 54.

Example 20.—A barrier before a ticket office in a works was constructed out of sheet metal, which was bent to the form of an ellipse of major axis 21 ins. and minor axis 5 ins. Find the area of sheet metal required if the height of the barrier is 5 ft.

In this case $a = 10{\cdot}5$ and $K = \dfrac{\sqrt{10{\cdot}5^2 - 2{\cdot}5^2}}{10{\cdot}5} = {\cdot}9714$

so that $K^2 = {\cdot}944$.

The table for the plotting reads—

ϕ	$\sin\phi$	$\sin^2\phi$	$1-K^2\sin^2\phi$	$\sqrt{1-K^2\sin^2\phi}$
0	0	0	1	1
·157	·1564	·0245	1 − ·0232 = ·977	·99
·314	·309	·0951	1 − ·0898 = ·91	·955
·471	·454	·206	1 − ·195 = ·805	·897
·628	·5878	·345	1 − ·326 = ·674	·821
·785	·7071	·5	1 − ·472 = ·528	·726
·942	·809	·652	1 − ·616 = ·384	·62
1·099	·891	·793	1 − ·749 = ·251	·501
1·256	·9511	·9	1 − ·85 = ·15	·388
1·413	·9877	·976	1 − ·922 = ·078	·279
1·570	1	1	1 − ·944 = ·056	·235

and the values in the extreme columns are plotted in Fig. 53.

The area under this curve = 1·0663 sq. unit

and thus the perimeter $= 4a \times 1{\cdot}0663 = 4 \times 10{\cdot}5 \times 1{\cdot}0663$
$= \underline{44{\cdot}78 \text{ ins.}}$

Hence the area required $= \dfrac{5 \times 44{\cdot}78}{12} = \underline{18{\cdot}66 \text{ sq. ft.}}$

It is well to compare this value of the perimeter with those obtained by the approximate rules given in Part I, p. 105—

(*a*) Perimeter $= \pi(a+b) = \pi(10{\cdot}5+2{\cdot}5) = 40{\cdot}84$ ins.

(*b*) Perimeter $= 4{\cdot}443\sqrt{a^2+b^2} = 4{\cdot}443 \times 10{\cdot}8 = 47{\cdot}96$ ins.

(*c*) Perimeter $= \pi\{1{\cdot}5(a+b) - \sqrt{ab}\} = \pi \times 14{\cdot}38 = 45{\cdot}16$ ins.

and the perimeter, correct to two places of decimals, is given in the tables * as 44·79 ins.

* The tables of complete elliptic integrals give the values of $\int_0^{\frac{\pi}{2}} \sqrt{1 - K^2\sin^2\phi}\, d\phi$ for various values of θ, θ being the angle whose sine is K, the eccentricity of the ellipse. Thus to use the tables for this particular case we put $\sin\theta = K = {\cdot}9714$, whence $\theta = 76° \, 16'$; we then read the values of the integral for 75°, 76° and 77°, and by plotting these values and interpolation we find that for the required

Thus the errors in the results found by the different rules are—

(*a*) 8·82 % too small (*b*) 7·08 % too large (*c*) ·83 % too large showing that the rule of Boussinesq gives an extremely good result in this case of a very flat ellipse, whilst the other approximate methods are practically worthless.

Area of Surface of a Solid of Revolution.—When a solid of revolution is generated, the boundary of the revolving figure sweeps out the surface of that solid. The volume of the solid depends upon the area of the revolving figure, whilst the surface depends upon the perimeter of the revolving figure.

To find the surface generated by the revolution of the curve CD about OX (Fig. 55) we must find the sum of the surfaces swept out by small portions of the curve, such as PQ. Let PQ = a small element of arc = δs. Then the outside surface of the solid generated by the revolution of the strip PQMN about OX will be equal to the circumference of the base × slant height, *i. e.*, $2\pi y\delta s$. Hence the total surface will be the sum of all similar elements, *i. e.*, $\sum_{x=a}^{x=b} 2\pi y\delta s$, approximately, or if δx becomes smaller and smaller—

$$\text{Surface} = \int_{x=a}^{x=b} 2\pi y ds.$$

For ds we may substitute its value, viz.—

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.\,dx$$

so that
$$\text{Surface} = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.\,dx.$$

---

angle, viz., 1·0664. Multiplication by $4a$, *i. e.*, 42, gives the result 44·79. For the convenience of readers interested in this question, and who desire a result more exact than that given by the approximate rules, a curve is here given (Fig. 54) with values of $\int_0^{\frac{\pi}{2}} \sqrt{1-K^2\sin^2\phi}\, d\phi$ plotted against values of K; and for the full perimeter of the ellipse the ordinates of this curve must be multiplied by twice the length of the major axis.

E. g., if the major axis = 16 and the minor axis = 10

$$K = \frac{\sqrt{8^2-5^2}}{8} = \cdot 7807.$$

Erecting an ordinate at K = ·7807 to meet the curve, we read the value 1·294; multiplying this by 32, we arrive at the figure 41·41, which is thus the required perimeter.

Example 21.—Find the area of the surface of a lune of a sphere of radius a, the thickness or height of the lune being b.

The surface will be that generated by the revolution of the arc CD of the circle about its diameter OX (Fig. 56).

From the figure $\qquad y^2 = a^2 - x^2$

whence $\qquad 2y \cdot \frac{dy}{dx} = -2x$

or $\qquad \frac{dy}{dx} = -\frac{x}{y}.$

Thus $\qquad \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{y^2} = \frac{y^2+x^2}{y^2} = \frac{\quad}{a^2-x^2}.$

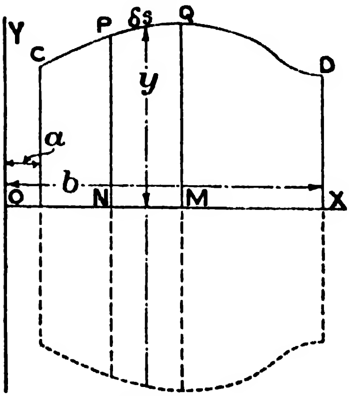


FIG. 55.

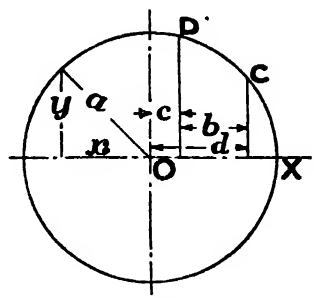


FIG. 56.

$$\text{Hence the surface} = \int_c^d 2\pi\sqrt{a^2-x^2}\,\frac{a}{\sqrt{a^2-x^2}}\,dx \qquad \{d = b+c\}$$
$$= 2\pi a\int_c^d dx$$
$$= 2\pi a(d-c) = \underline{2\pi ab.}$$

but $2\pi ab$ is the area of a portion of the lateral surface of the cylinder circumscribing the sphere.

Thus the surface of a lune of a sphere = the lateral surface of the portion of the cylinder circumscribing the sphere (the heights being the same).

Exercises 18.—On Volumes, Areas of Surfaces and Length of Arc.

1. The cross sections at various points along a cutting are as follows—

Distance from one end (ft.)	0	40	82	103	134	166	192	200
Area of cross section (sq. ft.)	0	210	296	295	244	154	50	0

Find the volume of earth removed in making the cutting.

2. Find the weight of the stone pillar shown in Fig. 57. The flanges are cylindrical, whilst the radius of the body at any section is determined by the rule, radius $= \frac{2}{\sqrt{x}}$, where x is the distance of the section from the fixed point O. (Weight of stone = 140 lbs. per cu. ft.)

3. The curve $y = 2x^2 - 3x$ revolves about the axis of x. Find the volume of the solid thus generated, the bounding planes being those for which $x = -2$ and $x = +4$.

4. Find, by integration, the surface of a hemisphere of radius r.

5. The curve $y = ae^{bx}$ passes through the points $x = 1$, $y = 3{\cdot}5$, and $x = 10$, $y = 12{\cdot}6$; find a and b. This curve rotates about the axis of x, describing a surface of revolution. Find the volume between the cross sections at $x = 1$ and $x = 10$.

6. Find the weight of a cylinder of length l and diameter D, the density of the material varying as the distance from the base. (Let the density of a layer distant x from base $= Kx$.)

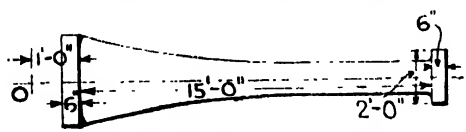


FIG. 57.—Weight of Stone Pillar.

7. The rectangular hyperbola having the equation $x^2 - y^2 = 25$ revolves about the axis of x. Find the volume of a segment of height 5 measured from the vertex.

8. The line $4y - 5x = 12$ revolves about the axis of x. Find the surface of the frustum of the cone thus generated, the limits of x being 1 and 5.

9. The radius of a spindle weight at various points along its length is given in the table—

Distance from one end (ins.)	0	·375	·5	1·0	1·3	1·6	1·85
Radius (ins.).	1·61	1·61	·78	·42	·4	·5	·5

Find its weight at ·283 lb per cu. in., the end portions being cylindrical.

10. Determine by the method indicated in *Example* 19, p. 204, the perimeter of an ellipse whose major axis is 30 ins. and whose minor axis is 18 ins. Compare your result with those obtained by the use of the approximate rules (*a*), (*b*) and (*c*) on p. 207.

11. The curve taken by a freely hanging cable weighing 3 lbs. per foot and strained by a horizontal pull of 300 lbs. weight conforms to the equation—

$$y = c \cosh \frac{x}{c}$$

where $c = \frac{300}{3}$. Find the total length of the cable if the span is 60 ft., *i. e.*, x ranges from -30 to $+30$.

12. Find the weight (in cast iron) of the centrifugal casting shown in Fig. 91A, p. 256.

Centre of Gravity and Centroid.—The Centre of Gravity (C. of G.) of a body is that point at which the resultant of all the forces acting on the body may be supposed to act, *i. e.*, it is the balancing point. The term *Centroid* has been applied in place of C. of G. when dealing with areas; and as our work here is more concerned with areas it will be convenient to adopt the term centroid.

From the definition it will be seen that the whole weight of a body may be supposed to act at its C. of G.; and in problems in Mechanics this property is most useful. Thus, movements of a complex system of weights may be reduced to the movement of the C. of G. of these. Or to take another instance: in structural work, in connection with fixed beams unsymmetrically loaded, it is necessary to find the position of the centroid of the bending-moment diagram. It is thus extremely important that rules should be found for fixation of the position of the centroid in all cases; and the methods adopted may be divided into two classes: (*a*) algebraic (including purely algebraic, and partly algebraic and partly graphic), (*b*) graphic.

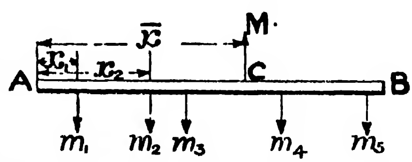


Fig. 58.—Centre of Gravity or Centroid.

The rules will best be approached by way of a simple example on moments. In place of areas or solids, afterwards to be dealt with, let us consider the case of a uniform bar loaded as shown in Fig. 58.

For equilibrium the two conditions to be satisfied are—

(1) The upward forces balance the downward forces.

(2) The right-hand moments about any point balance the left-hand moments about the same point; or, in other words, the algebraic sum of the moments about any point is zero.

If C is the balancing point or fulcrum, the upward reaction of the fulcrum $= M = m_1+m_2+m_3+m_4+m_5$ from condition (1).

Taking moments about A, let $\bar{x}$ (x bar) be the distance AC.

Then, by condition (2)—

$$M\bar{x} = m_1x_1+m_2x_2+m_3x_3+ \ldots$$

or

$$\bar{x} = \frac{m_1x_1+m_2x_2+m_3x_3+ \ldots}{m_1+m_2+m_3+ \ldots}$$

$$= \frac{\Sigma mx}{\Sigma m}.$$

The product of a force into its distance from a fixed point or axis is called its *first moment* about that point or axis; whilst the

product of a force into the square of its distance from a fixed point is called its *second moment* about that point.

Hence our statement concerning the distance AC can be written—

$$\bar{x} = \frac{\Sigma \text{ 1st moments}}{\Sigma \text{ masses}}.$$

To extend this rule to meet the case of a number of scattered masses arranged as in Fig. 59, the co-ordinates of the centroid must be found, viz., $\bar{x}$ and y.

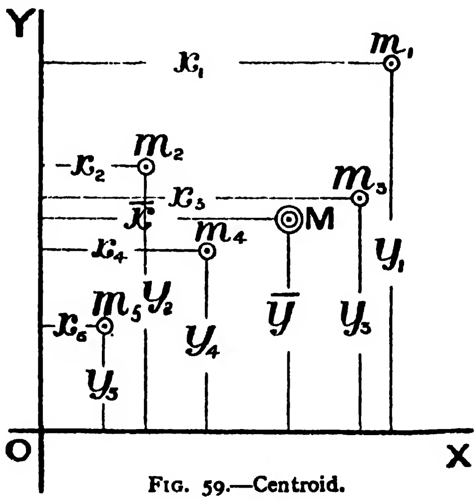


FIG. 59.—Centroid.

Thus $$\bar{x} = \frac{\Sigma mx}{\Sigma m} = \frac{\Sigma \text{ 1st moments about OY}}{\Sigma \text{ masses}}$$

and $$\bar{y} = \frac{\Sigma my}{\Sigma m} = \frac{\Sigma \text{ 1st moments about OX}}{\Sigma \text{ masses}}.$$

If the masses are not all in one plane, their C. of G. must be found by marking their positions in a plan and elevation drawing and determining the C. of G. of the elevations and also that of their plan. Thus the C. of G. is located by its plan and elevation.

It will be observed that some form of summation is necessary for the determination of the positions of centroids or centres of gravity; and this summation may be called by a different name, viz., integration, all the rules of which may be utilised; the integration in some cases being graphic, in some cases algebraic, and in others a combination of the two.

Rules for the Determination of the Centroid of an Area.—Let it be required to find the centroid of the area ABCD in Fig. 60.

The area may be considered to be composed of an infinite number of small elements or masses, each being the mass of some thin strip like PQMN; the co-ordinates of the centre of gravity of which may be determined in the manner already explained.

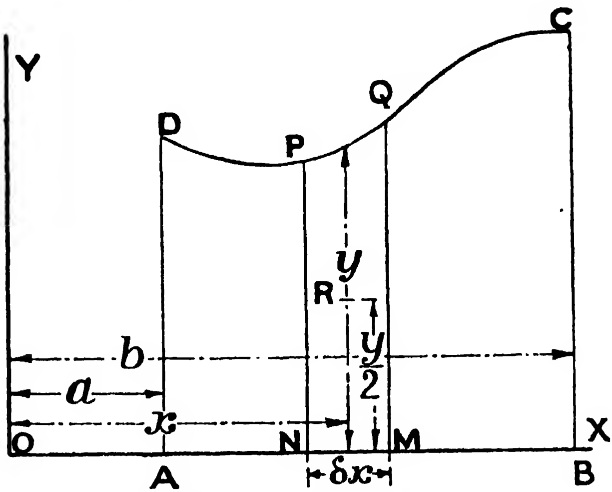


FIG. 60.—Centroid of an Area.

To find $\bar{x}$, *i. e.*, the distance of the centroid from OY—

Mass of strip PQMN = area × density (considering the strip as of unit thickness)

$$= y\delta x \times \rho$$

1st moment of strip about OY = mass × distance $= \rho y \delta x \times x$

$$= \rho x y \delta x.$$

Hence $$\bar{x} = \frac{\Sigma \text{ 1st moments about OY}}{\Sigma \text{ masses}}$$

$$\simeq \frac{\Sigma\, \rho x y \delta x}{\Sigma\, \rho y \delta x} \quad \text{the limits being } a \text{ and } b$$

and if the strips are made extremely narrow—

$$\bar{x} = \frac{\int_a^b \rho x y dx}{\int_a^b \rho y dx} = \frac{\int_a^b x y dx}{\int_a^b y dx}$$

ρ cancelling from both numerator and denominator.

Thus a vertical is found on which the centroid of the area must lie; and this line is known as the centroid vertical.

To fix the actual position of the centroid some other line must be drawn, say a horizontal line, the intersection of which with the centroid vertical is the centroid.

Thus the height of the centroid above OX must be found; this being denoted by $\bar{y}$.

To find $\bar{y}$.—The whole mass of the strip PQMN may be supposed to act at R, its mid-point, because the strip is of uniform density; and hence the moment of the strip PQMN about OX

$$= \text{mass} \times \text{distance} = \rho y\,\delta x \times \frac{y}{2}$$

$$= \frac{\rho}{2} y^2 \delta x.$$

Hence
$$\bar{y} = \frac{\Sigma_a^b \text{ 1st moments about OX}}{\Sigma_a^b \text{ masses}}$$

$$= \frac{\int_a^b \frac{\rho}{2} y^2 dx}{\int_a^b \rho y\, dx} = \frac{1}{2}\frac{\int_a^b y^2 dx}{\int_a^b y\, dx}$$

As in previous cases, the integration may be algebraic, this being so when y is stated in terms of x, or graphic, when a curve or values of y and x are given.

Suppose the latter is the case, and we desire to find $\bar{x}$—

Then
$$\bar{x} = \frac{\int xydx}{\int ydx}$$

and the values of the numerator and denominator must be found separately. Each of these gives the area of a figure, for if Y is written in place of xy, the numerator becomes $\int Ydx$, which is the standard expression for the area under the curve in which Y is plotted against x; and the denominator is already in the required form.

Thus a new set of values must be calculated, viz., those of Y, these being obtained by multiplication together of corresponding values of x and y; and these values of Y are plotted to a base of x. Then the area under the curve so obtained is the value of the numerator, and the denominator is the area under the curve with y plotted against x; and, finally, division of the one by the other fixes the value of $\bar{x}$.

Example 22.—Find the centroid of the area bounded by the curve given by the table, the axis of x and the ordinates through $x = 10$ and $x = 60$.

x	10	25	40	45	50	60
y	4	5·3	6·2	6·4	6·6	6·8

We thus wish to find the centroid of the area ABCD (Fig. 61). To find $\bar{x}$:—

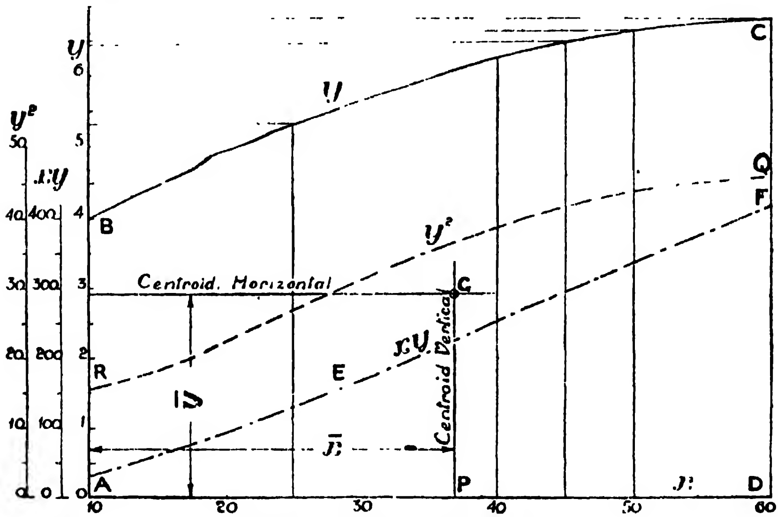


FIG. 61.—Centroid of an Area.

The table for the plotting of Y against x reads—

x . .	10	25	40	45	50	60
Y or xy	40	132·4	248	288	330	408

From this we get the curve AEF.

The area of the figure ABCD—

$$i.\ e.,\qquad \int_{10}^{60} y dx = 289$$

and the area of the figure AEFD—

$$i.\ e.,\qquad \int_{10}^{60} xy dx = 10650$$

The method of integration is not shown, to avoid confusion of curves.

$$\therefore\qquad \bar{x} = \frac{\int_{10}^{60} xy dx}{\int_{10}^{60} y dx} = \frac{10650}{289} = \underline{36{\cdot}9.}$$

Thus the centroid vertical, or the line PG is fixed.

We need now to find the centroid horizontal, *i.e.*, $\bar{y}$ must be determined.

$$\text{Now}\qquad \bar{y} = \frac{\frac{1}{2}\int_{10}^{60} y^2dx}{\int_{10}^{60} ydx} = \frac{\frac{1}{2}\int_{10}^{60} \mathrm{Y}\,dx}{\text{area of ABCD}} \quad \left\{\begin{matrix}\text{where Y in this case}\\ \text{stands for } y^2\end{matrix}\right\}$$

so that the following table must be compiled—

x . .	10	25	40	45	50	60
Y or y^2 .	16	28	38·4	41	43·5	46·1

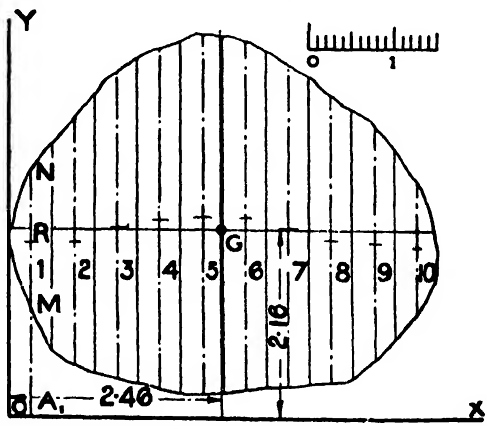


FIG. 62.—C. of G. of Thin Plate.

Plotting from this table, the curve RQ results, and the area of the figure ARQD is 1689.

$$\therefore \quad \bar{y} = \frac{\frac{1}{2}\text{ area of ARQD}}{\text{area of ABCD}} = \frac{\frac{1}{2}\times 1689}{289} = \underline{2 \cdot 92.}$$

The intersection of the centroid vertical and the centroid horizontal at G fixes the centroid of ABCD.

A modification of this method is necessary when the actual area is given in place of the tabulated list of values, the procedure being outlined in the following example.

Example 23.—It is required to find the C. of G. of a thin plate having the shape shown in Fig. 62. Show how this may be done.

Draw two convenient axes at right angles and divide up the area into thin strips by lines drawn parallel to OY. Draw in, also, the mid-ordinates of these strips. The area of any strip can be assumed

to be "mean height × thickness"; and therefore measure ordinates such as MN and multiply by the thickness or width of the strip. Repeat for each strip, and the sum of all these will be the area of the figure.

To find $\bar{x}$.—OA_1 = the distance of the centre of 1st strip from OY so that the area of strip × OA_1 = 1st moment of strip about OY.

Hence, multiply the area of each strip by the distance of its mid-ordinate from OY and add the results; then the sum will be the 1st moment of the area about OY.

Then $$\bar{x} = \frac{\text{Sum of 1st moments}}{\text{Area}} = \frac{\text{2nd total}}{\text{1st total}}.$$

To find $\bar{y}$.—Fix R, the mid-point of MN, and do the same for all the strips. The area of the strip has already been found; multiply this by A_1R and repeat for all strips. The sum of all such will be the 1st moment about OX; dividing this by the area of the figure, the distance, $\bar{y}$, of the centroid from OX is found.

[Note that R is the mid-point of MN and not of NA_1, because OX is a purely arbitrary axis.]

For this example the calculation would be set out thus—

Strip	Length of mid-ordinate (like MN)	Width of Strip	Area of Strip	Distance of centre from OY (like OA_1)	Distance of centre from OX (like RA_1)	1st moment about OX	1st moment about OY
1	1·55	·5	·775	·25	2·0	1·55	·19
2	2·79	·5	1·395	·75	2·0	2·79	1·05
3	3·44	·5	1·720	1·25	2·18	3·75	2·05
4	3·85	·5	1·925	1·75	2·27	4·37	3·36
5	4·01	·5	2·005	2·25	2·32	4·64	4·50
6	3·92	·5	1·960	2·75	2·3	4·51	5·40
7	3·60	·5	1·800	3·25	2·18	3·92	5·85
8	3·26	·5	1·630	3·75	2·04	3·32	6·11
9	2·66	·5	1·330	4·25	2·02	2·68	5·65
10	1·47	·5	·735	4·75	1·95	1·43	3·49
	Totals		15·275			32·96	37·65

$$x = \frac{37{\cdot}65}{15{\cdot}28} = 2{\cdot}46$$

and $$\bar{y} = \frac{32{\cdot}96}{15{\cdot}28} = 2{\cdot}16.$$

Thus the position of G is fixed by the intersection of a horizontal at a height of 2·16 with a vertical 2·46 units distance from OY.

If the centroid of an arc was required, the lengths of small elements of arc would be dealt with in place of the small areas, but otherwise the procedure would be the same.

"Double Sum Curve" Method of Finding the Centroid Vertical.—This method is convenient when only the centroid vertical is required; for although entirely graphic, it is rather too long to be used for fixing the centroid definitely.

Method of Procedure.—To find the centroid vertical for the area APQH (Fig. 63).

Sum curve the curve PQ in the ordinary way, thus obtaining the curve A*eg*E; for this construction the pole is at O, and the polar distance is p.

Produce PA to O_1, making the polar distance p_1 = HE = last ordinate of the sum curve of the original curve (viz., PQ).

Sum curve the curve A*eg*E from AP as base and with O_1 as

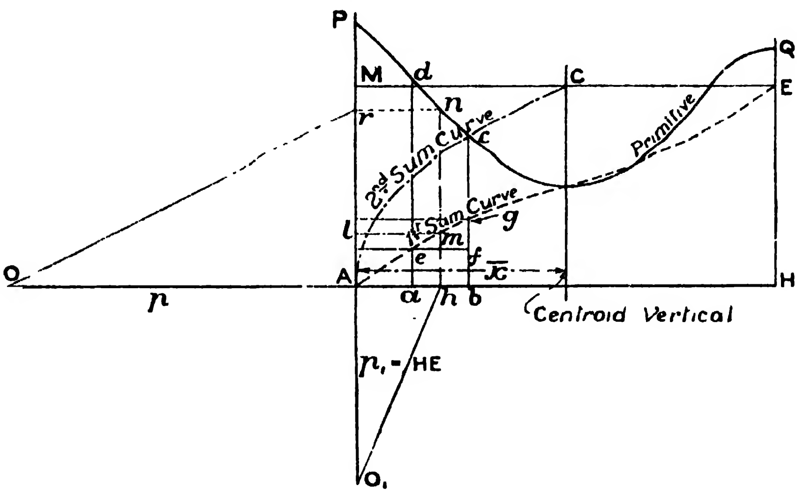


FIG. 63.—Centroid Vertical of an Area.

pole; then the last ordinate of this curve, viz., CM, is of length $\bar{x}$, so that the vertical through C is the centroid vertical.

Proof.—Consider the strip *abcd*, a portion of the original area. Then O*r* and *eg* are parallel (by construction)—

and thus $$\frac{p}{Ar} = \frac{ef}{fg} = \frac{ab}{fg}$$

or $$Ar \times ab = p \times fg$$

i. e., $$hn \times ab = p \times fg$$

$$\therefore \quad \Sigma hn \times ab = \Sigma p \times fg = p\Sigma fg = p \, . \, \mathrm{HE}.$$

Again, the 1st moment of the strip about AP = area × distance

$$= hn \times ab \times Ah$$
$$= hn \times ab \times ml$$
$$= p \times fg \times ml$$

and hence 1st moment of area APQH about AP—

$$= \Sigma p \times fg \times ml = p\Sigma fg \times ml$$
$$= p \times p_1 \times CM$$

but 1st moment of area APQH about AP—

$$= \text{area} \times \text{distance of centroid from A}$$
$$= p \times HE \times x$$

$\therefore \quad p \times p_1 \times CM = p \times HE \times x \quad$ (also $p_1 = HE$)

and $\quad x = MC.$

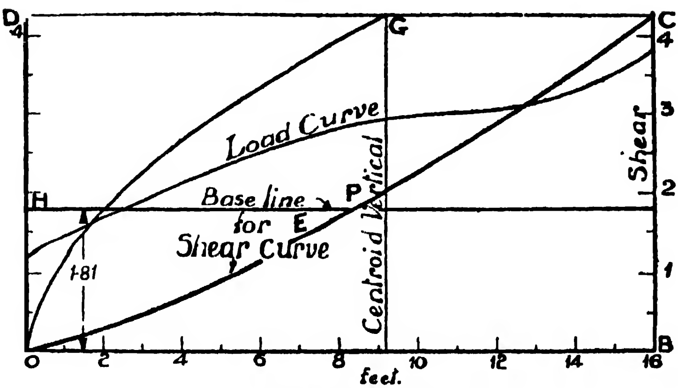


FIG. 64.—Problem on Loaded Beam.

Example 24.—A beam, 16 ft. long, simply supported at its ends is loaded with a continuously varying load, the loading being as expressed in the table.

Distance from left-hand support (feet)	0	2	4	6	8	10	12	14	16
Load in tons per foot run	·12	·17	·21	·25	·28	·29	·31	·34	·38

Find the centroid vertical of the load curve, and hence determine the reactions of the supports and the point at which the maximum bending moment occurs.

We first plot the load curve from the figures given in the table (Fig. 64); and next we sum curve this curve, taking a polar distance of 10 horizontal units; the last ordinate of this sum curve reads 4·27, so that the total load is 4·27 tons. We now set off AD equal in length to BC, and with this as polar distance we sum curve the curve AEC from the vertical axis as base. This sum curve finishes at the point G on the horizontal through C, and a vertical through G is the centroid vertical, distant 9·2 ft. from the end A.

For purposes of calculation, the whole load may be supposed to

act in this line; the total load is 4·27 tons, and taking moments round A—

$$4{\cdot}27 \times 9{\cdot}2 = R_B \times 16$$

whence $\qquad R_B = 2{\cdot}46$ tons

and $\qquad R_A = 4{\cdot}27 - 2{\cdot}46 = 1{\cdot}81$ tons.

We now set up AH, a distance to represent R_A, to the new vertical scale, and then a horizontal through H is the true base line of shear.

At the point P the shear is zero; but the shear is measured by the rate of change of bending moment, so that zero shear corresponds to maximum bending moment; and hence, grouping our results—

Reaction at left-hand support $= \underline{1{\cdot}81 \text{ tons}}$

Reaction at right-hand support $= \underline{2{\cdot}46 \text{ tons}}$

and the maximum bending moment occurs at a distance of $\underline{8{\cdot}4 \text{ ft.}}$ from the left-hand end.

Centroids of Sections by Calculation (for a graphic method especially applicable to these, see p. 251).—Special cases arise in

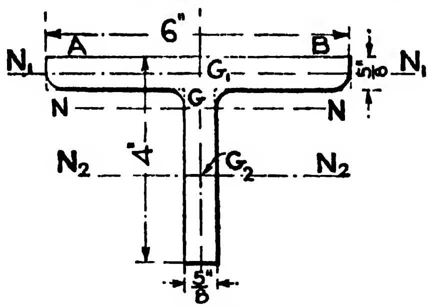


Fig. 65.

the form of sections of beams, joists, rails, etc., for which a modification of the previous methods is sufficient.

If the section is composed of a combination of simple figures, such as rectangles or circles, as in the great majority of cases it is, its centroid can be found by loading each of its portions into which for purposes of calculation it may be divided, with a weight proportional to its area, and treating the question as one for the determination of the C. of G. of a number of isolated weights.

Example 25.—Find the position of the centroid of the Tee section shown in Fig. 65.

We may consider the section to be made up of two rectangles; then—

$$\text{Area of flange} = 6 \times \frac{5}{8} = \frac{30}{8} \text{ sq. ins.} = \frac{240}{64} \text{ sq. ins.}$$

and the centroid of the flange is at G_1.

Area of web $\quad 3\frac{3}{8}\times\frac{5}{8} \quad \frac{135}{64}$ sq. ins.

and the centroid of the web is at G_2.

From considerations of symmetry we see that the centroid of the section must lie on the line G_1G_2, at the point G, say.

Treat G_1G_2 (of length $\frac{5''}{16}+\frac{27''}{16}$ *i.e.*, 2″) as a bar loaded with $\frac{240}{64}$ units at G_1 and $\frac{135}{64}$ units at G_2.

Let $G_1G = \bar{x}$; then the upward force at $G = \frac{240}{64}+\frac{135}{64}$

$$= \frac{375}{64} \text{ units.}$$

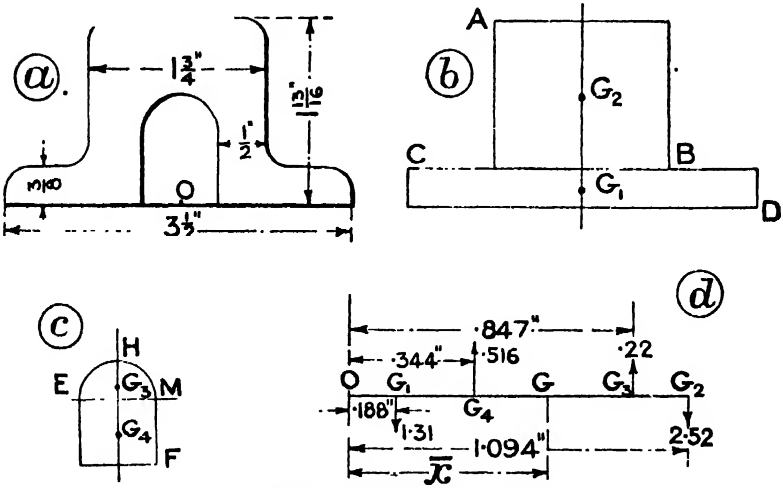


FIG. 66.—Centroid of Bridge Rail.

(In the further calculation we may disregard the denominators, since they are alike.)

Taking moments about G_1

$$375\times\bar{x} = 135\times 2$$

whence $\qquad \bar{x} = \frac{270}{375} = \cdot 72.$

Hence the distance of the centroid from the outside of the flange—

$$= \frac{5''}{16} + \cdot 72'' = \underline{1\cdot 03''}.$$

Example 26.—Determine the position of the centroid of the bridge rail section shown at (*a*), Fig. 66.

This example presents rather more difficulty than the one immediately preceding it. The plan of procedure is, for cases such as this,

that adopted in the work on the calculation of weights, viz., we first treat the section as "solid" and then subtract the part cut away.

Neglecting the small radii at the corners, and treating the section as "solid," the section has the form shown at (*b*), Fig. 66.

The area of AB $= \frac{23}{16} \times \frac{7}{4} = 2{\cdot}52$ sq. ins., and its centroid is at G_2, the intersection of its diagonals.

Similarly the area of CD $= \frac{3}{8} \times \frac{7}{2} = 1{\cdot}313$ sq. ins., and its centroid is at G_1.

For the part cut away (see (*c*), Fig. 66)

The area of EHM $= \frac{\pi}{2} \times \left(\frac{3}{8}\right)^2 = {\cdot}221$ sq. in.; and we know from Part I, p. 130, that its centroid G_3 is distant $\cdot 424 \times$ radius, *i. e.*, $\cdot 424 \times \cdot 375$ or $\cdot 159''$ from EM.

Again, the area of EF $= \frac{11}{16} \times \frac{3}{4} = {\cdot}516$ sq. in., and its centroid is at G_4.

Our problem is thus reduced to that of determining the C. of G. of four isolated weights, two of which act in the direction opposed to that of the others, placed as shown at (*d*), Fig. 66.

Let the centroid of the whole section be at G, distant $\bar{x}$ from O.

Now the upward forces = the downward forces

$$\text{and thus} \qquad R_G + {\cdot}516 + {\cdot}221 = 2{\cdot}52 + 1{\cdot}313$$

$$\text{whence} \qquad R_G = 3{\cdot}096.$$

Also, by taking moments about O—

$$(3{\cdot}096 \times \bar{x}) + ({\cdot}516 \times {\cdot}344) + ({\cdot}221 \times {\cdot}847) = (1{\cdot}313 \times {\cdot}188) + (2{\cdot}52 \times 1{\cdot}094)$$

$$\text{whence} \qquad \bar{x} = {\cdot}855 \text{ in.}$$

or the centroid of the section is $\underline{\cdot 855''}$ distant from the outside of the flange.

Centroids found by Algebraic Integration.—Suppose that the equation of the bounding curve is given, then the centroid of the area between the curve, the axis and the bounding ordinates may be determined by algebraic integration.

We have already seen that—

$$\bar{x} = \frac{\int xy\,dx}{\int y\,dx} \text{ and } \bar{y} = \frac{\frac{1}{2}\int y^2 dx}{\int y\,dx}$$

so that if y is stated as a function of x, xy and y^2 may be expressed in terms of x, and the integration performed according to the rules given.

The examples here given should be carefully studied, for there

are many possibilities of error arising due to the incorrect substitution of limits.

Example 27.—Find the centroid of the area between the curve $y = 2x^{1\cdot5}$, the axis of x and the ordinates through $x = 2$ and $x = 5$.

The curve is plotted in Fig. 67, and it is seen that the position of the centroid of the area ABCD is required.

Now $\qquad y = 2x^{1\cdot5}$, and thus $xy = 2x^{\frac{3}{2}} \times x = 2x^{\frac{5}{2}}$
and $y^2 = 4x^3$.

To find $\bar{x}$—

$$\bar{x} = \frac{\int_2^5 xy\,dx}{\int_2^5 y\,dx} = \frac{\int_2^5 2x^{\frac{5}{2}}dx}{\int_2^5 2x^{\frac{3}{2}}dx}$$

$$= \frac{\left(\frac{2}{7}x^{\frac{7}{2}}\right)_2^5}{\left(\frac{2}{5}x^{\frac{5}{2}}\right)_2^5}$$

$$= \frac{\frac{2}{7}\times\frac{5}{2}\left\{5^{\frac{7}{2}}-2^{\frac{7}{2}}\right\}}{\left\{5^{\frac{5}{2}}-2^{\frac{5}{2}}\right\}}$$

$$= \frac{5}{7}\times\frac{268}{50\cdot25}$$

$$= 3\cdot81$$

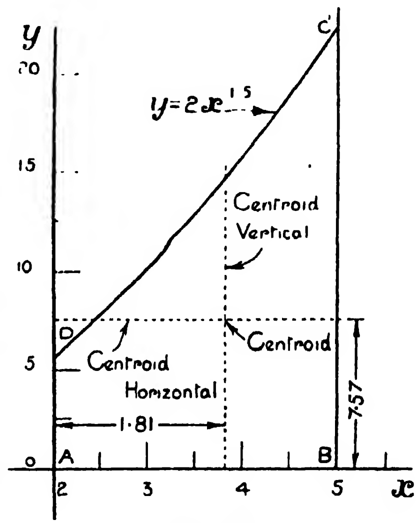


FIG. 67.

or the centroid vertical is distant 1·81 units from the left-hand boundary.

To find $\bar{y}$—

$$\bar{y} = \frac{\frac{1}{2}\int_2^5 y^2dx}{\int_2^5 y\,dx}$$

$$= \frac{\frac{1}{2}\times 4\int_2^5 x^3dx}{2\int_2^5 x^{\frac{3}{2}}dx} = \frac{\left(\frac{x^4}{4}\right)_2^5}{\left(\frac{2}{5}x^{\frac{5}{2}}\right)_2^5}$$

$$= \frac{1}{4}\times\frac{5}{2}\,\frac{(5^4-2^4)}{\left(5^{\frac{5}{2}}-2^{\frac{5}{2}}\right)}$$

$$= \frac{5}{8}\times\frac{609}{50\cdot25} = 7\cdot57 \text{ units.}$$

Hence the co-ordinates of the centroid are 3·81, 7·57.

Example 28.—The bending moment curve for a beam fixed at one end and loaded uniformly over its whole length is a parabola, as

shown in Fig. 68. The vertex is at A and the ordinate at B, viz., BC is $\frac{wl^2}{2}$; the loading being w units per foot and l being the span.

We wish to determine the position of the centroid of the figure ABC so that we may find the moment of the area ABC about AD, and finally the deflection at A.

From the equation to a parabola, $y^2 = 4ax$, we see that—

$$l^2 = 4a.\frac{wl^2}{2}, \text{ whence } 4a = \frac{2}{w} \text{ or } y^2 = \frac{2}{w}x$$

$$i.e., \quad (\mathrm{ND})^2 = \frac{2}{w}\,\mathrm{AD}.$$

The distance of the centroid from AD = y

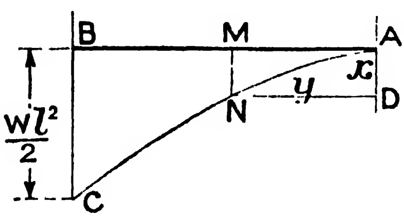


FIG. 68.

$$= \frac{\int_0^l xy\,dy}{\int_0^l x\,dy}$$

$$= \frac{\int_0^l \frac{w}{2}y^3 dy}{\int_0^l \frac{w}{2}y^2 dy} = \frac{\left(\frac{1}{4}y^4\right)_0^l}{\left(\frac{1}{3}y^3\right)_0^l}$$

$$= \underline{\frac{3}{4}l.}$$

Area of ABCD $= \frac{1}{3}$ of surrounding rectangle $= \frac{1}{3}\times l \times \frac{wl^2}{2}$

$$= \frac{wl^3}{6}.$$

All this area may be supposed to be concentrated at its centroid, and hence the moment of ABC about AD $= \frac{wl^3}{6}\times\frac{3}{4}l = \frac{wl^4}{8}$.

Now the deflection at A $= \frac{1}{\mathrm{EI}}\times$ moment of the bending moment diagram about the vertical through A

$$= \frac{1}{\mathrm{EI}}\times\frac{wl^4}{8}.$$

Hence the deflection at A $= \frac{wl^4}{8\mathrm{EI}} = \underline{\frac{\mathrm{W}l^3}{8\mathrm{EI}}}$, where W = total load.

Example 29.—Find the position of the centroid of a quadrant of a circle of radius r

The equation of the circle is $x^2+y^2 = r^2$

hence $\qquad y = \sqrt{r^2-x^2}$

so that $\qquad xy = x\sqrt{r^2-x^2}.$

Thus $\bar{x}$ (and consequently $\bar{y}$)—

$$=\frac{\int_0^r x\sqrt{r^2-x^2}\,dx}{\int_0^r \sqrt{r^2-x^2}\,dx}$$

The value of the denominator is $\frac{\pi r^2}{4}$, for it is the area of the quadrant. (This integral would be evaluated as shown on p. 149.)

To evaluate the numerator, let $u = r^2-x^2$

then $$du = -2xdx$$

or $$xdx = -\frac{du}{2}.$$

Then—

$$\int_0^r x\sqrt{r^2-x^2}\,dx = \int_{x=0}^{x=r} -\frac{du}{2}u^{\frac{1}{2}}$$

$$= -\frac{1}{2}\Big(\frac{2}{3}u^{\frac{3}{2}}\Big)_{x=0}^{x=r}$$

$$= -\frac{1}{3}\Big[(r^2-x^2)^{\frac{3}{2}}\Big]_0^r$$

$$= -\frac{1}{3}[0-(+r^2)^{\frac{3}{2}}]$$

$$= \frac{1}{3}r^3.$$

$$\therefore\quad \bar{x} = \bar{y} = \frac{\frac{1}{3}r^3}{\frac{\pi r^2}{4}} = \frac{4r}{3\pi} \quad\text{or}\quad \cdot 424r.$$

Centre of Gravity of Irregular Solids.—The methods given for the determination of the centroids of irregular areas apply equally well when solids are concerned. For if A is the area of the cross section of a solid at any point along its length, distant x, say, from one end, and the length is increased by a small amount δx (and if this is small there will be no appreciable change in the value of A), then the increase in the volume $= A\delta x$ or the increase in the weight $= \rho A\delta x$, ρ being the density.

The moment of this element about the end $= \rho A\delta x \times x$

so that $$\bar{x} = \frac{\Sigma \text{ 1st moments}}{\Sigma \text{ weights}} = \frac{\int_0^l \rho Axdx}{\int_0^l \rho Adx}$$

$$= \frac{\int_0^l Axdx}{\int_0^l Adx}$$

As before, two cases arise, viz., (*a*) when values of A and x are given, and (*b*) when A is defined in terms of x. To deal with these—

In case (*a*) plot one curve in which A is the ordinate and x is the abscissa and find the area under it; this is the value of $\int_0^l A dx$.

Plot a second curve whose ordinates are the products of corresponding values of A and x and find the area; this is the value of the numerator, and division of the latter area by the former gives the value of $\bar{x}$. Thus the centroid vertical is found, and if the solid is symmetrical about the axis of x, this is all that is required; otherwise the centroid horizontal must be found, the procedure being exactly that previously described when dealing with areas in place of volumes.

An example on the application of this method is here worked.

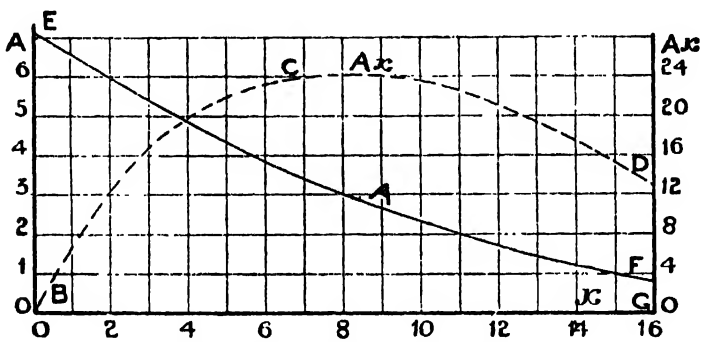


FIG. 69.—Problem on C.I. Column.

Example 30.—The circumference of a tapering cast iron column, 16 ft. long, at 5 equidistant places is 9·43, 7·92, 6·15, 4·74 and 3·16 ft. respectively. Find its volume and the distance of its C. of G. from the larger end.

The areas must first be found from the circumferences.

Now the area of a circle $= \frac{(\odot ce)^2}{4\pi}$

So that the table for plotting reads—

x = distance from larger end (ft.)	0	4	8	12	16
A = area of cross section (sq. ft.)	7·09	4·98	3·0	1·78	·79

By plotting these values the curve EF (Fig. 69) is obtained.

The figure here given is a reproduction of the original drawing to rather less than half-size, and since the measurements were made on the original, the results now stated refer to that.

In the original drawing the scales were: 1" vertically = 2 sq. ft., and 1" horizontally = 2 ft., so that 1 sq. in. of area represented 4 cu. ft. of volume. The area under the curve EF was found, by means of the planimeter, to be 13·66 sq. ins., and accordingly the volume = $13 \cdot 66 \times 4$ = 54·64 cu. ft.

The curve BCD results from the plotting of values of Ax as ordinates, the table for which plotting reads—

x	0	4	8	12	16
Ax	0	19·9	24	21·4	12·6

The area under this curve was found to be 19·06 sq. ins., which

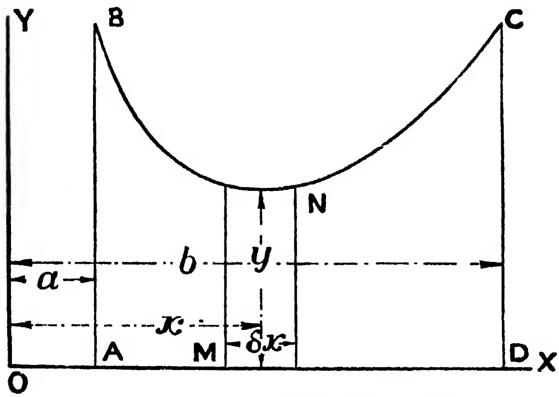


FIG. 70.—C. of G. of Solid of Revolution.

represented $19 \cdot 06 \times 16$ units of moment, since for the plotting of BCD 1" vertically = 8 units of Ax, and 1" horizontally = 2 units of x.

Hence $$\bar{x} = \frac{\text{area BCDG}}{\text{area BEFG}} = \frac{16 \times 19 \cdot 06}{54 \cdot 64} = 5 \cdot 58 \text{ ft.}$$

For case (*b*), when A is stated in terms of x, the integration is entirely algebraic. Thus if A is a function of x, integrate Ax and also A with regard to x, and divide the former integral by the latter to determine the value of $\bar{x}$.

Example 31.—The area of cross section of a rod of uniform density varies as the cube root of the distance of the section from one end; find the distance of the C. of G. from that end, being given that the area at a distance x from the end = $4 \cdot 5\sqrt[3]{x}$.

Consider a strip distant x from the stated end and of thickness δx.

Then, from hypothesis, the area of section $= 4.5\sqrt[3]{x}$, and thus the volume $=$ area $\times$ thickness $= 4.5\sqrt[3]{x}\times\delta x$.

Also the mass of the strip $=$ volume $\times$ density

$$= 4.5\sqrt[3]{x}\times\delta x\times\rho$$

and the moment of the strip about the end $=$ mass $\times$ distance

$$= 4.5\rho x^{\frac{1}{3}}\delta x\times x$$
$$= 4.5\rho x^{\frac{4}{3}}\delta x.$$

Hence
$$\bar{x} = \frac{\Sigma \text{ 1st moments of small elements}}{\Sigma \text{ their masses}}$$

$$= \frac{\int_0^l 4.5\rho x^{\frac{4}{3}}dx}{\int_0^l 4.5\rho x^{\frac{1}{3}}dx} = \frac{\int_0^l x^{\frac{4}{3}}dx}{\int_0^l x^{\frac{1}{3}}dx}$$

$$= \frac{3}{7}\times\frac{4}{3}\left(\frac{l^{\frac{7}{3}}}{l^{\frac{4}{3}}}\right)$$

$$= \frac{4l}{7}$$

or the C. of G. is distant $\frac{4}{7}$ of the length from the given end.

C. of G. of a Solid of Revolution.—Suppose that the curve BC in Fig. 70 rotates round OX as axis; and we require to find the position of the C. of G. of the solid so generated.

Consider a small strip of area MN; its mean height is y and its width is δx, so that the volume generated by the revolution of this is $\pi y^2\delta x$, or the mass $= \rho\pi y^2\delta x$. The 1st moment of this strip about OY $=$ mass $\times$ distance $= \rho\pi y^2\delta x\times x = \rho\pi xy^2\delta x$.

Thus the total 1st moment about OY $\triangleq \sum_a^b \rho\pi xy^2\delta x$

and the total mass $\triangleq \sum_a^b \rho\pi y^2\delta x$

$$\bar{x} = \frac{\int_a^b \rho\pi xy^2dx}{\int_a^b \rho\pi y^2dx} = \frac{\int_a^b xy^2dx}{\int_a^b y^2dx}.$$

As before, the two cases arise, viz.—

(*a*) When values of x and y are given. For this case make a table of values of $x\times y^2$ and also one of values of y^2.

Plot the values of xy^2 against those of x and find the area under the resulting curve

$$\text{This area} = \int xy^2dx \quad \ldots\ldots \quad (1)$$

Plot the values of y^2 against those of x—

$$\text{Area of figure so obtained} = \int y^2 dx \quad . \quad . \quad . \quad (2)$$

and
$$\bar{x} = \frac{(1)}{(2)}.$$

Also we know that y must be zero, for the axis of x is the axis of rotation; and thus the C. of G. is definitely fixed.

(*b*) When y is expressed as a function of x. In this case find both xy^2 and also y^2 in terms of x, integrate these functions algebraically and thence evaluate the quotient.

Example 32.—The curve given by the tabulated values of y and x revolves about the x-axis; find the position of the C. of G. of the solid thus generated.

x	0	1	2	3	4
y	8	10	21	26·4	25

For the first curve, values of xy^2 are required, and for the second curve, values of y^2; these values being—

x	0	1	2	3	4
y^2	64	100	441	696	625
xy^2	0	100	882	2088	2500

The curve AB (Fig. 71) is obtained by plotting the values of xy^2 as ordinates; and the area under this curve is 4323; this being thus the value of $\int_0^4 xy^2 dx$.

By plotting the values of y^2 as ordinates the curve CD is obtained; and the area under this curve is 1699, *i. e.*, $\int_0^4 y^2 dx = 1699$.

Then—
$$\bar{x} = \frac{\int_0^4 xy^2 dx}{\int_0^4 y^2 dx} = \frac{4323}{1699} = 2{\cdot}55 \text{ units}$$

i. e., the C. of G. is at G, the point (2·55, 0).

Example 33.—The curve $x = 5y - 2\sqrt{y}$ revolves about the axis of y. Find the position of the centre of gravity of the solid generated, the solid being bounded at its ends by the horizontal planes distant 1 and 5 units respectively from the axis of x.

Since the revolution is about the axis of y and not that of x, y must take the place of x in our formulæ and x the place of y; therefore the limits employed must be those for y.

In Fig. 72 AB is the curve $x = 5y - 2\sqrt{y}$, and we see that it is

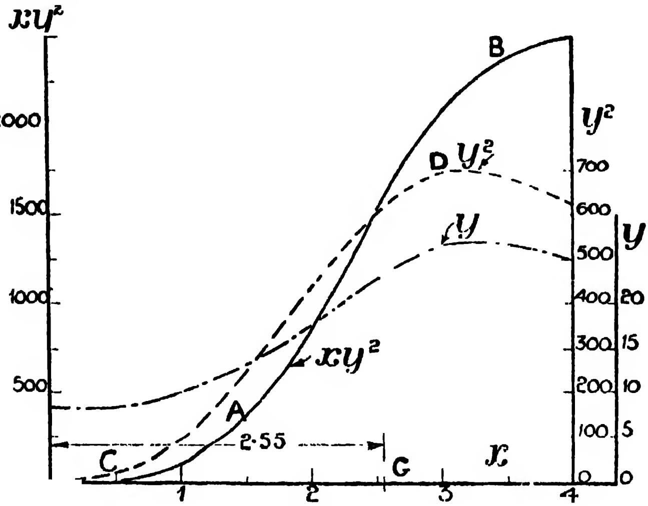


FIG. 71.

required to find the height of the centroid above the axis of x of the solid generated by the curve AB about the axis of y.

Then to find $\bar{y}$—

$$\bar{y} = \frac{\int_1^5 yx^2dy}{\int_1^5 x^2dy}$$

Now $\quad x = 5y - 2\sqrt{y}$, and thus $x^2 = 25y^2 + 4y - 20y^{\frac{3}{2}}$

and $\quad yx^2 = 25y^3 - 20y^{\frac{5}{2}} + 4y^2$.

Then $$\int_1^5 yx^2dy = \int_1^5 (25y^3 - 20y^{\frac{5}{2}} + 4y^2)\,dy = \left[\frac{25y^4}{4} - \frac{20\times 2}{7}y^{\frac{7}{2}} + \frac{4y^3}{3}\right]_1^5$$

$$= \left(\frac{25\times 625}{4}\right) - \left(\frac{40}{7}\times 280\right) + \left(\frac{4\times 125}{3}\right) - \frac{25}{4} + \frac{40}{7} - \frac{4}{3}$$

$$= 2471$$

and $$\int_1^5 x^2dy = \int_1^5 (25y^2 + 4y - 20y^{\frac{3}{2}})\,dy = \left[\frac{25y^3}{3} + \frac{4y^2}{2} - \frac{20\times 2}{5}y^{\frac{5}{2}}\right]_1^5$$

$$= \left(\frac{25\times 125}{3}\right) + (2\times 25) - (8\times 55{\cdot}9) - \frac{25}{3} - 2 + 8$$

$$= 642$$

$$\therefore \qquad y = \frac{\int_1^5 yx^2dy}{\int_1^5 x^2dy} = \frac{2471}{642} = \underline{3{\cdot}85}.$$

Then since the centroid must lie along the axis of y, its position is definitely fixed at the point G, viz., (0, 3·85).

Example 34.—Find the mass and also the position of the C. of G. of a bar of uniform cross section a and length l, whose density is proportional to the cube of the distance from one end.

Let us consider a small length δx of the bar, distant x from the end mentioned above; the density of the material here $= Kx^3$, where K is some constant; hence—

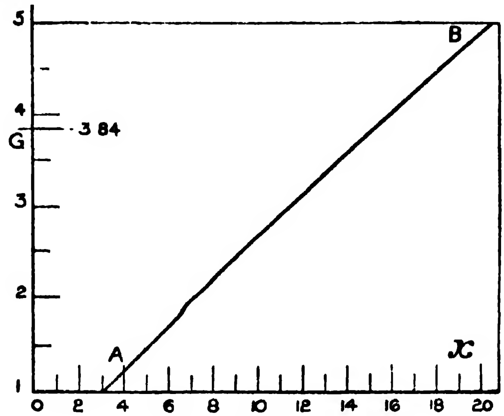


FIG. 72.

Mass of small element = volume × density $= a\delta x \times Kx^3 = Kax^3\delta x$

Thus the total mass $= \int_0^l Kax^3dx = Ka\left(\frac{x^4}{4}\right)_0^l$

$$= \frac{Kal^4}{4}.$$

Also the 1st moment of the element about the end—

$$= \text{mass} \times \text{distance}$$

$$= Kax^3\delta x \times x.$$

$\therefore$ Total 1st moment $= \int_0^l Kax^4dx = \frac{Kal^5}{5}$

and if $\bar{x}$ = distance of C. of G. from the lighter end—

$$\bar{x} = \frac{\frac{Kal^5}{5}}{\frac{Kal^4}{4}} = \frac{4}{5}l.$$

Example 35.—Find the position of the C. of G of a triangular lamina whose density varies as the distance from the apex. (Let the thickness of the lamina $= t$.)

Consider a small strip of width δx, distant x from the apex (Fig. 73).

The area of the strip $= y\delta x$, and thus its volume $= yt\delta x$.

Now the density $\propto x$ or density $= Kx$

and also, by similar triangles, $\frac{y}{x} = \frac{B}{H}$

or $y = \frac{Bx}{H}$

So that the mass of the strip $= yt\delta x \times Kx$

$$= \frac{BKt}{H}x^2\delta x$$

and the 1st moment of the strip about OY—

$$= \frac{BKt}{H}x^3\delta x.$$

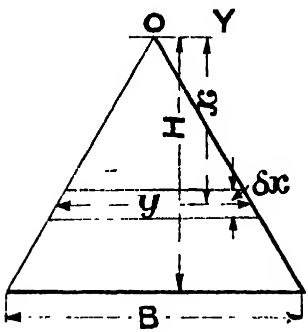


FIG. 73.

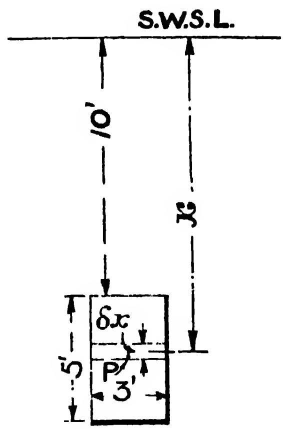


FIG. 74.

Hence—

$$\bar{x} = \frac{\int_0^H \frac{BKt}{H}x^3dx}{\int_0^H \frac{BKt}{H}x^2dx}$$

$$= \frac{\left(\frac{x^4}{4}\right)_0^H}{\left(\frac{x^3}{3}\right)_0^H} = \frac{H^4}{4} \times \frac{3}{H^3}$$

$$= \underline{\frac{3}{4}H.}$$

Centre of Pressure.—If a body is immersed in a liquid, then the pressure per sq. in. of surface is not uniform over the solid, for the pressure is proportional to the depth The point at

which the total pressure may be supposed to act is known as the *centre of pressure* (C. of P.).

To find positions of centres of pressure we are, in effect, finding centres of gravity of solids whose density is proportional to the distance from some fixed axis.

The C. of G. found in the example last worked is in reality the C. of P. of a triangular lamina immersed vertically in a liquid, with OY as the level of the top of the liquid.

Just as, when discussing the stability of solids in air, we have supposed the whole mass to be concentrated at the C. of G., so now, when the solid is immersed in a liquid, the total pressure may be assumed to act at the one point, viz., the C. of P.

To find the positions of the C. of P. for various sections and solids we must start from first principles, dealing with the pressure on small elements, and then summing.

Example 36.—Find the whole pressure on one side of a rectangular sluice gate of depth 5 ft. and breadth 3 ft., if the upper edge is 10 ft. below the level of the water (which we shall speak of as the still water surface level or S.W.S.L). Find also the depth of the centre of pressure.

Consider a strip of the gate δx deep and x ft. below S.W.S.L. (Fig. 74).

Then the area of the strip $= 3 \times \delta x$

and the pressure per sq. ft $=$ K $\times$ depth.

Now at a depth of x ft. the pressure per sq. ft. $=$ weight of a column of water x ft. high and 1 sq. ft. in section, *i. e.*, wt. of x cu. ft. of water or $62{\cdot}4x$ lbs.

Also the pressure is the same in all directions;

and thus the pressure on the strip $= 3\delta x \times 62{\cdot}4x$

and the moment of the pressure on the strip about S.W.S.L.—

$$= 187{\cdot}2x\delta x \times x$$
$$= 187{\cdot}2x^2\delta x.$$

Hence the total pressure $= \int_{10}^{15} 187{\cdot}2x\,dx$ lbs.

$$= 187{\cdot}2\left(\frac{x^2}{2}\right)_{10}^{15}$$
$$= \frac{187{\cdot}2 \times 125}{2} \text{ lbs.}$$
$$= 11700 \text{ lbs. or } \underline{5{\cdot}23 \text{ tons.}}$$

Again, the total 1st moment about S.W.S.L.—

$$= \int_{10}^{15} 187{\cdot}2x^2dx = 187{\cdot}2\left(\frac{x^3}{3}\right)_{10}^{15}$$
$$= 62{\cdot}4 \times 2375.$$

Thus the depth of the C. of P. below S.W.S.L.—

$$= \frac{62 \cdot 4 \times 2375}{11700} \text{ ft.}$$
$$= 12 \cdot 65 \text{ ft.}$$

Hence C. of P. is at the point P, at a depth of 12·65 ft. below the surface of the liquid.

The more general investigation for the position of the C. of P. is given in Chap. X.

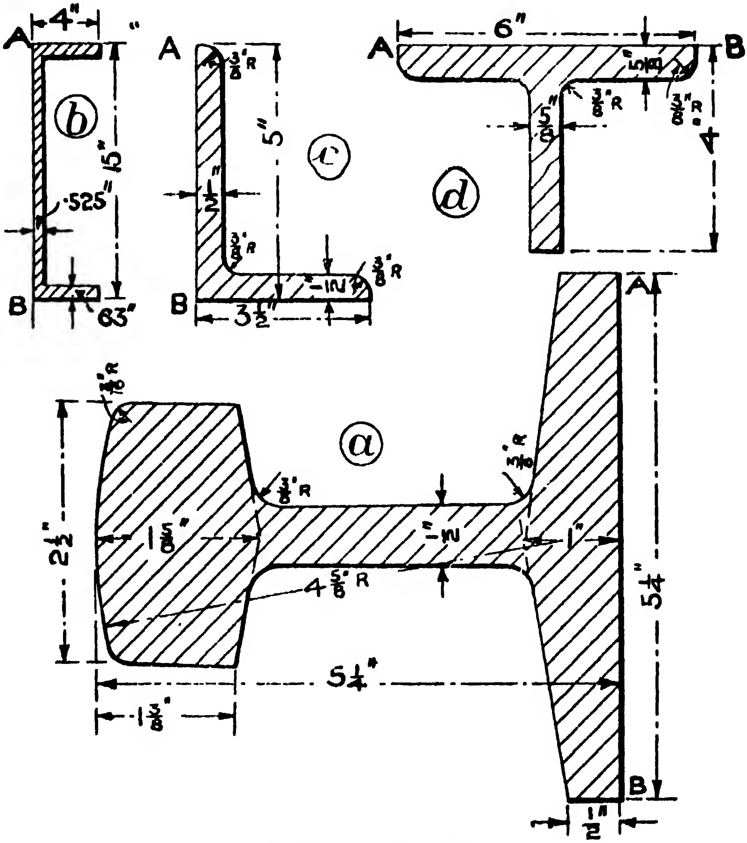


FIG. 75.—Centroids.

Exercises 19.—On the Determination of the Positions of Centroids and Centres of Gravity.

1. The density of the material of which a right circular cone is composed varies as the square of the distance from the vertex. Find the position of the centre of gravity of the cone.

2. The equidistant half-ordinates of the load water plane of a ship are as follows, commencing from forward: ·6, 2·85, 9·1, 15·54, 18, 18·7, 18·45, 17·6, 15·13 and 6·7 ft. respectively. Find the area of the load water plane and the longitudinal position of its centroid. The length of the ship on the load water line is 270 ft.

3. A triangular plate of base 5″ and height 8″ is immersed in water, its base being along the S.W.S.L. Find the total pressure on the plate and the depth of the centre of pressure if the plate is vertical.

4. A vertical retaining wall is 8 ft. wide and 15 ft. deep. Find the depth of the centre of pressure of the earth on the wall.

5. Draw the quadrant of a circle of 4″ radius, and by the double sum curve method determine the position of its centroid.

6. The portion of the parabola $y = 2x^2 - 9x$ below the x axis revolves about that axis. Find the volume of the paraboloid so generated, and the distance of its C. of G. from the axis of y.

7. Find the position of the centroid of the area bounded by the curve $y = 1{\cdot}7 - {\cdot}2x^2$, the axis of x and the ordinates through $x = -1$ and $x = +4$.

8. Reproduce (*a*), Fig. 75, to scale (full size), and find the position of the centroid of the section represented, employing the method outlined in *Example* 23, p. 216.

9. Draw a segment of a circle of diameter = 6″ on a base of 5·92″, and find by the method of *Example* 23, p. 216, the height of the centroid above the base. (Take the segment that is less than a semicircle.)

Find the distance of the centroid from the line AB for the sections in Nos. 10, 11 and 12.

10. Channel Section, (*b*), Fig. 75.

11. Unequal Angle, (*c*), Fig. 75.

12. Tee Section, (*d*), Fig. 75.

13. Make a careful drawing of (*a*) Fig. 76, which represents the half-section of the standard form of a stream line strut for an aeroplane, taking t as 2″, and by the method of *Example* 23, p. 216, determine the distance of the centroid from the leading edge.

14. Find the position of the centroid of the pillar shown in Fig. 57, p. 210, of which further explanation is given in Question 2 on p. 210. [Deal with the flanges and the body as three separate portions.]

15. One end of a horizontal water main 3 ft. in diameter is closed by a vertical bulkhead, the centre of the main being 35 ft. below the level of the water. Find the total pressure on the bulkhead.

16. A semicircular plate is immersed vertically in sea water, its diameter being along the water surface. Find the total pressure on the plate if its diameter is 12 ft. and the weight of 1 cu. ft. of sea water is 64 lbs.; find also the depth of the centre of pressure. [*Note.*—The reduction formulæ given on p. 178 assist in the evaluation of the integrals.]

17. The parabola $y^2 = 6x$ revolves about the axis of x. Find the distance from the vertex of the C. of G. of the paraboloid thus generated, if the diameter of the end of the paraboloid is 18.

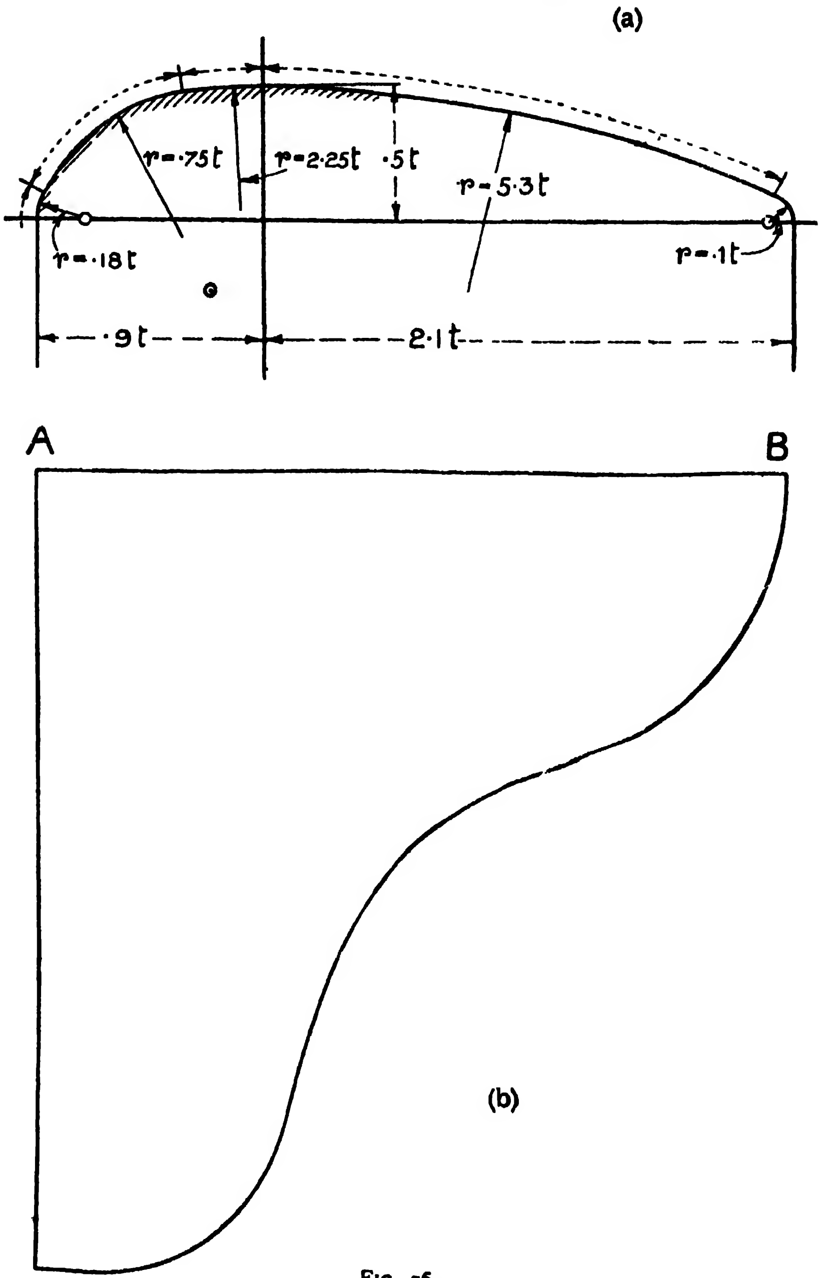


FIG. 76.

18. The diameter of a spindle at various distances along its length was measured with the following results—

Distance from end (ins.)	0	1	2	3	4	5	6	7	8
Diameter (ins.) . . .	1·5	1·12	·83	·85	1·18	1·5	1·78	1·96	2

Find the distance of the C. of G. from the smaller end.

19. Find, by means of the double sum curve method, the distance from AB of the centroid of the rail section shown at (*a*), Fig. 75.

20. An aluminium right circular cone is of height 7 ins. and the diameter of its base is 10 ins. Find (*a*) its mass, the density of aluminium being ·093 lb. per cu. in.; (*b*) the height of its C. of G. above the base.

21. Use the double sum curve method to find the distance from AB of the centroid of the area shown at (*b*), Fig. 76.

22. A segment of a parabola is of height h and stands on a base b. Find the height of the centroid above the base.

23. A triangular plate of height h is immersed in water, its vertex being at the water surface, and its base being horizontal. Find the depth of the centre of pressure of the plate.

Moment of Inertia.—The product of a mass into the square of its distance from some fixed point or axis is called its second moment about that point or axis; and for a number of masses the sum of their respective second moments becomes the second moment, or *moment of inertia* of the system. When the number of masses is infinite, *i. e.*, when they merge into one mass, the limiting value of the sum of the second moments is spoken of as the moment of inertia of the body.

The moment of inertia of a section or body determines to a large extent the strength of the section or body to resist certain strains; the symbol I, which always stands for moment of inertia, occurs in numerous engineering formulæ; also when dealing with the formulæ of angular movement the mass is replaced by I, and so on, so that it is extremely important that one should be able to calculate values of I for various sections or solids.

A few examples will emphasise the frequent recurrence of the letter I. Consider first the case of a loaded beam:—

Let the figure (Fig. 77) represent the section of a beam loaded in any way. Then it is customary to make the following assumptions—

(*a*) There is to be no resultant stress over the section, *i. e.*, the sum of the tensions = the sum of the compressions.

(*b*) That the stress varies as the strain, and that the Young's modulus for the material is the same for tension as for compression.

(*c*) That the original radius of curvature of the beam is exceedingly great compared with the dimensions of the cross section of the beam.

The surface of the beam which is neither compressed nor stretched is spoken of as the *neutral surface*, and the line in which this cuts any cross section of the beam is known as the *neutral axis*.

Referring to Fig. 77, let NN be the neutral axis, and let σ be

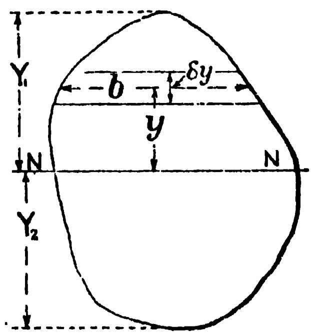


FIG. 77.

the stress at unit distance from NN, *i. e.*, $\sigma y =$ the stress at a distance y from NN.

Thus the stress at y on a section of breadth b and depth $\delta y = \sigma y$, and the force $=$ stress $\times$ area $= b\delta y \times \sigma y$.

Now the forces on one side of NN must balance those on the other (by hypothesis).

$$\therefore \qquad \int_{-Y_2}^{Y_1} bdy\sigma y = 0.$$

$$\text{but} \qquad \int_{-Y_2}^{Y} \sigma bdy \times y = \text{total 1st moment of the forces}$$

and the line about which this is zero must pass through the centroid of the section; hence the line NN passes through the centroid.

The tensile and compressive forces form a couple, the moment of which—

$$= \Sigma \text{ force} \times \text{distance} \doteq \sum_{-Y_2}^{Y_1} b\delta y\sigma y \times y$$

$$\doteq \sigma \sum_{-Y_2}^{Y_1} b\delta y \times y^2$$

i. e., in the limit the moment of resistance of the internal forces

$$= \sigma \int_{-Y_2}^{Y_1} bdy \times y^2, \quad i.\ e.,\ \sigma \int \text{area} \times (\text{distance})^2$$

$$i.\ e.,\ \sigma\ (\text{2nd moment of section about NN})$$
$$= \sigma I.$$

If M is the bending moment at the section, *i. e.*, the moment of the external forces, it must be exactly balanced by the moment of the internal forces, so that $M = \sigma I$.

Also if f_1 = maximum tensile stress and $= \sigma Y_1$

f_2 = maximum compressive stress and $= \sigma Y_2$

then $$\sigma = \frac{f_1}{Y_1} = \frac{f_2}{Y_2} = \frac{M}{I}$$

or, in general, $$\frac{M}{I} = \frac{f}{y}$$

Hence, in considering the strength of a beam to resist bending, it is necessary to know the moment of inertia of its section; knowing this and the bending moment, we can calculate the maximum skin stress.

As a further illustration of the importance of I in engineering formulæ let us deal with the following case: If a magnet is allowed to swing in a uniform field, the time T of a complete oscillation is given by—

$$T = 2\pi \sqrt{\frac{I}{MH}}$$

where I = moment of inertia of the magnet

M = magnetic moment of the magnet

H = strength of the uniform field in which the magnet swings.

In this case the I of a cuboid is required; and it will be seen that no mention of the mass is made in this formula. Actually the I takes account not only of the mass, but also of its disposition, the latter being a most important factor in all questions of angular movement. Thus for a mass of 1 lb. swinging at the end of an arm of 10 ft. the energy would be 10^2, *i. e.*, 100 times that of the same mass placed at a radius of 1 ft. only, although the angular velocities in the two cases were the same.

The reason for the presence of I in formulæ concerning the energy of rotation will be better understood if the next Example is carefully studied.

Example 37.—A disc revolves at n revs. per sec.; find an expression for its energy of rotation, or its kinetic energy.

If the total mass $= m$, let a small element δm of mass be considered, distant r from the axis of rotation (Fig. 78).

Now the linear velocity at the rim $= V = 2\pi n R$

and the angular velocity $= \omega =$ number of radians per sec.

$$= 2\pi n$$

then $$R\omega = 2\pi n R = V$$

or $$\omega = \frac{V}{R}.$$

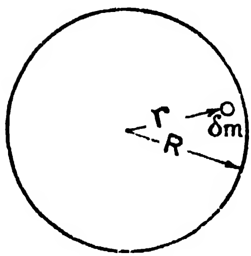


FIG. 78.

thus ω is constant throughout, whilst V depends on the radius.

Kinetic Energy of mass δm

$$= \frac{\text{mass} \times (\text{veloc.})^2}{2} = \frac{\delta m \times v^2}{2}$$

$$= \frac{1}{2}\omega^2 r^2 \delta m$$

$$= \frac{\omega^2}{2} r^2 \delta m.$$

Hence the total K.E. of the disc $= \int_0^R \frac{\omega^2}{2} r^2 dm$

$$= \frac{\omega^2}{2}\int_0^R \text{mass} \times (\text{distance})^2$$

$$= \frac{\omega^2}{2} \times I \text{ for disc.}$$

Thus the K.E. $= \frac{1}{2} I\omega^2$. Comparing this formula with the corresponding one for linear motion, *viz.*, K.E. $= \frac{1}{2} mv^2$, we see that when changing from linear to angular movement, I *takes the place of m and ω the place of v.*

Suppose that the average velocity $= v_1 = r_1\omega$

then $$\frac{1}{2} mv_1^2 = \frac{1}{2} I\omega^2$$

i. e., $$mr_1^2\omega^2 = I\omega^2$$

or $$I = mr_1^2.$$

Hence I is of the nature of mass $\times$ (distance)^2, so that if the whole mass were concentrated at the distance r_1 from the axis, the K.E. of the system would be unaltered.

Hence the distance r_1 (which is usually denoted by k) is referred to as the *swing or spin radius*, or *radius of gyration, i. e.*, it is the effective radius as regards all questions of rotation.

[Note that k is not the arithmetic mean of the various radii, but the R.M.S. value for—

$$k = \sqrt{\frac{\Sigma\,(\text{radius})^2}{\text{number considered}}}.]$$

In general, I can be written as mk^2 (if dealing with a mass) or Ak^2 (if concerned with an area).

Method of Determination of the Value of I for any Section.—Whilst it is found desirable to commit to memory the values of I for the simpler sections, it is not wise to trust entirely to this plan. It is a far better policy to understand thoroughly the meaning of the term "moment of inertia," and to derive its value for any section or solid by working directly from first principles.

Thus, knowing that the moment of inertia is obtained by summing up a series of second moments, we divide the area or mass into a number of very small elements, find the area or mass of each of these and multiply each area or mass by the square of its distance from the axis or point about which moments are required; the sum of all such products being the value of I.

If the length of the swing radius is required, it can be determined from the relation $I = Ak^2$ (for an area) or $I = Mk^2$ (for a solid); the area or mass being obtained by the summation of the areas or the masses of the separate elements.

Thus $$k = \sqrt{\frac{\Sigma \text{ second moments of elements}}{\Sigma \text{ areas or masses of elements}}}.$$

Confusion often arises over the units in which I is measured; and to avoid this it is well to think of I in the form Ak^2 or Mk^2, when it is observed that I is of the nature area $\times$ (length)^2, *i. e.*, (inches)$^2 \times$ (inches)^2 or (inches)^4 for a section, and mass $\times$ (length)^2 or lbs. $\times$ (inches)^2 for a solid.

The moment of inertia must always be expressed with regard to some particular axis; and it is frequently necessary to change from one axis to another. To assist in this change of axis the following rules are necessary:—

The Parallel Axis Theorem.—By means of this theorem, if I is known about an axis through the C. of G., the I about an axis parallel to the first can be deduced.

In Fig. 79 NN is the neutral axis of the section; and the moment of inertia is required about AB, *i. e.* I_{AB} is required.

Dealing with the strip indicated—

$$I_{AB} \text{ of the strip} = \rho b\delta y \times y^2.$$

$$\text{Hence the total } I_{AB} = \rho\int bdy \times y^2$$
$$= \rho\int bdy \times (Y-d)^2$$
$$= \rho\int bdy(Y^2+d^2-2Yd)$$
$$= \rho\int bdy \times Y^2 + \rho\int bdy \times d^2 - 2\rho\int bdy Yd.$$

Now $\int \rho bdy \times Y^2 =$ the total I_{NN}

and $\int \rho bdy \times d^2 = d^2\int \rho bdy = d^2 \times$ total mass $= md^2$

also $2d\int \rho bdy \times Y = 2d \times$ total 1st moment about NN

$= 2d \times 0$ (for the moments on the strips on one side of NN balance those on the other)

$= 0.$

Hence— $I_{AB} = I_{NN} + md^2$

i. e., **to find the moment of inertia about any axis, find the moment of**

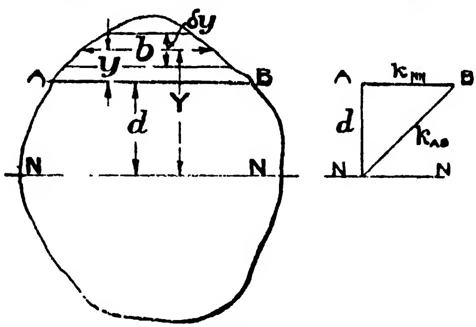


Fig. 79.

inertia about an axis through the C. of G. parallel to the axis given, and to this add the product of the mass into the square of the distance between the axes.

e. g., if $I_{NN} = 47$. mass $= 12{\cdot}4$ and d (between AB and NN$=2{\cdot}3$)

then $I_{AB} = I_{NN} + md^2 = 47 + (12{\cdot}4 \times 2{\cdot}3^2)$

$= 47 + 65{\cdot}6 = 112{\cdot}6.$

Since $I_{AB} = I_{NN} + md^2$

then $mk^2_{AB} = mk^2_{NN} + md^2$

or $k^2_{AB} = k^2_{NN} + d^2$

and this relation is represented by Fig. 79, which suggests a graphic method of finding k_{AB} when k_{NN} is known.

Theorem of Perpendicular Axes.—We require to find I about an axis perpendicular to the plane of the paper and passing through O; such being spoken of as a *polar second moment.*

To distinguish between the moment about an axis perpendicular to the plane of the paper and that about an axis in the paper, we shall adopt the notation I_O for the former and I_{OX} or I_{OY}, as the case may be, for the latter.

To find I_O :—

Consider a small element of mass δm at P (Fig. 80).

Then I_{OX} of this element $= \delta m \times y^2$, $I_{OY} = \delta m \times x^2$,

and $I_O = \delta m \times r^2$.

Now $\qquad r^2 = x^2+y^2$

hence $\qquad \delta m \,.\, r^2 = \delta m \,.\, x^2+\delta m \,.\, y^2$

and $\qquad \int dm \,.\, r^2 = \int dm \,.\, x^2+\int dm \,.\, y^2$

i. e., $\qquad$ total I_O = total I_{OY}+total I_{OX}

or $\qquad I_O = I_{OY}+I_{OX}$

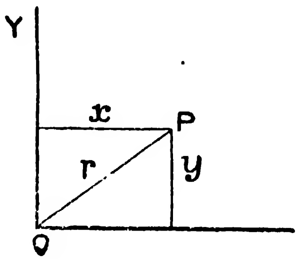


FIG. 80.

so that **if the moments of inertia about two perpendicular axes in the area are known, the sum of these is the moment of inertia about an axis perpendicular to the area and through the point of intersection of these axes.**

In special cases for which $\qquad I_{OX} = I_{OY}$

then $\qquad I_O = 2I_{OX}.$

To find the Relation between the Moment of Inertia about a Point in a Solid Body and the Moments of Inertia about three mutually Perpendicular Axes meeting in that Point.

Thus, referring to Fig. 81, it is desired to connect I_O with I_{OX}. I_{OY} and I_{OZ}.

Consider a small element of the mass δm placed at the point F

Then if $\qquad PS = x \quad PT = y \quad PM = z \quad OP = r$

$(ON)^2+(NM)^2+(PM)^2 = (OP)^2$, and $\quad ON = PS$, $NM = PT$

i. e., $\qquad x^2+y^2+z^2 = r^2.$

Now I_{OX} of the element $= \delta m(PN)^2 = \delta m(z^2+y^2)$

and in like manner $I_{OY} = \delta m(x^2+z^2)$ and $I_{OZ} = \delta m(y^2+x^2)$

also $\qquad I_O = \delta m(OP)^2 = \delta m r^2.$

Thus

$$I_O = \delta m r^2 = \delta m(x^2+y^2+z^2)$$

$$= \delta m\left(\frac{x^2+y^2+y^2+z^2+z^2+x^2}{2}\right)$$

$$= \delta m \times \frac{1}{2}(I_{OZ}+I_{OX}+I_{OY})$$

And for the total mass—

$$\int \delta m r^2 = \frac{1}{2}\int \delta m(I_{OX}+I_{OY}+I_{OZ})$$

or $$\text{total } I_O = \frac{1}{2}(I_{OX}+I_{OY}+I_{OZ}).$$

We may now apply the principles already enunciated to the determination of the moments of inertia of various sections and

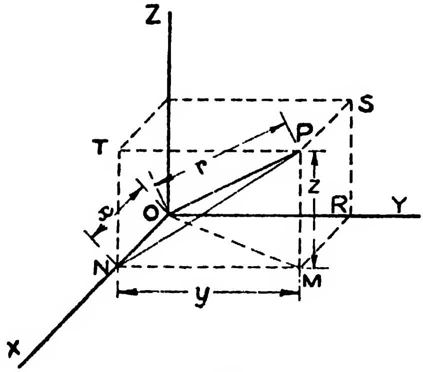


FIG. 81.

solids; and we take as our first example the case of a rectangular section.

Example 38.—To find the moments of inertia of a rectangle about various axes.

(*a*) To find I_{NN} (Fig. 82), NN being the neutral axis.

Dealing with the small strip, of thickness δx—

$$I_{NN} \text{ of strip} = b\delta x \times x^2 \quad i.e., \text{ area} \times (\text{distance})^2$$

Hence the total $$I_{NN} = \int_{-\frac{h}{2}}^{\frac{h}{2}} bx^2dx = b\left(\frac{x^3}{3}\right)_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{bh^3}{12}$$

$$= bh \times \frac{h^2}{12}$$

$$= \text{area} \times \frac{h^2}{12}$$

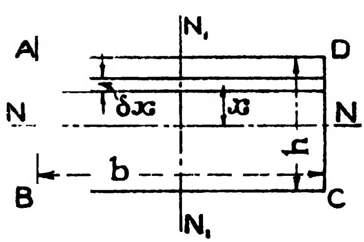


FIG. 82.

but $I_{NN} = A \times (k_{NN})^2$,

where A is the area of the section

thus $$A\frac{h^2}{12} = A(k_{NN})^2$$

or $$(k_{NN})^2 = \frac{h^2}{12}.$$

By symmetry it will be seen that $I_{N_1N_1} = A \times \frac{b^2}{12}$.

(*b*) To find I_{AB}.

$$I_{N_1N_1} = \frac{Ab^2}{12}$$

and the distance between AB and $N_1N_1 = \frac{b}{2}$

hence $$I_{AB} = I_{N_1N_1} + \left\{A \times \left(\frac{b}{2}\right)^2\right\} = A\frac{b^2}{12} + A\frac{b^2}{4}$$
$$= \frac{Ab^2}{3}.$$

I_{AB} is larger than $I_{N_1N_1}$, as would be expected, for the effective radius must be greater if the plate swings about AB than if it swings about N_1N_1.

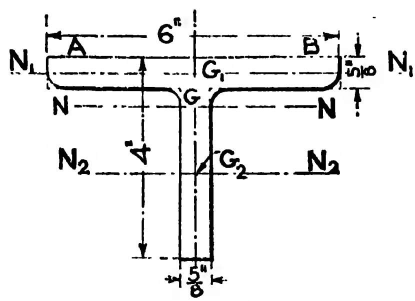


FIG. 83.

In like manner— $$I_{AD} = \frac{Ah^2}{3}$$

$$\therefore \quad I_A = I_{AB} + I_{AD} = A\frac{b^2}{3} + A\frac{h^2}{3}$$
$$= \frac{A}{3}(h^2+b^2) = \frac{1}{3}A \times (BD)^2.$$

The rule for the moment of inertia of a rectangle is required very frequently, since many sections can be broken up into rectangles.

Example 39.—To find I_{NN} of the Tee section shown in Fig. 83.

The neutral axis NN is distant 1·03" from AB (cf. *Example* 25, p. 220). Dealing with the flange—

$$I_{N_1N_1} = \frac{1}{12}bh^3 = \frac{1}{12} \times 6 \times \left(\frac{5}{8}\right)^3 = \cdot 122 \text{ in.}^4$$

also the distance $G_1G = \cdot 72"$.

Hence by the parallel axis theorem—

I_{NN} of the flange $= I_{N_1N_1} + [A \times (G_1G)^2]$ (A being the area of the flange)

$$= \cdot 122 + (6 \times \tfrac{5}{8} \times \cdot 72^2) = \cdot 122 + 1 \cdot 94$$
$$= 2 \cdot 06 \text{ ins.}^4$$

For the web $\quad I_{N_2N_2} = \frac{1}{12} bh^3 = \frac{1}{12} \times \left(\frac{27}{8}\right)^3 \times \frac{5}{8} = 2 \text{ ins.}^4$

and also $\quad G_2G = 1 \cdot 28''$.

Hence I_{NN} of the web $= I_{N_2N_2} + A_2 \times (G_2G)^2$ (A_2 is area of the web)

$$= 2 + \left(\frac{27}{8} \times \frac{5}{8} \times 1 \cdot 28^2\right)$$
$$= 5 \cdot 45 \text{ ins.}^4$$

Hence the total I_{NN} of the section $= 2 \cdot 06 + 5 \cdot 45 = \underline{7 \cdot 51 \text{ ins.}^4}$

Example 40.—Find the polar 2nd moment of a circular disc of radius R; and also the moment of inertia about a diameter.

Consider a ring of width δr, distant r from the centre (Fig. 84).

Then $\quad I_O$ of the annulus $=$ area $\times$ (distance)^2

$$= 2\pi r \delta r \times r^2$$
$$= 2\pi r^3 \delta r.$$

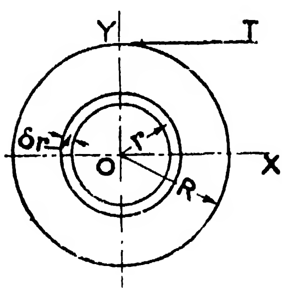


FIG. 84.

Hence

the total $I_O = \int_0^R 2\pi r^3 dr = 2\pi \int_0^R r^3 dr$

$$= \frac{2\pi R^4}{4} = \underline{\frac{\pi R^4}{2}}.$$

Now $\quad I_{OX} = I_{OY}$

and $\quad I_O = I_{OX} + I_{OY} = 2I_{OX}$

$\therefore \quad I_{OX} = \frac{1}{2} I_O = \underline{\frac{\pi R^4}{4}}.$

To find the respective swing radii—

$$Ak_O^2 = I_O = \frac{\pi R^4}{2}$$

$\therefore \quad \pi R^2 k_O^2 = \frac{\pi R^4}{2}$

i. e., $\quad k_O^2 = \frac{R^2}{2}$

or $\quad k_O = \frac{R}{\sqrt{2}}$, *i. e.*, $\underline{\cdot 707 R.}$

Also $\quad Ak_{OX}^2 = I_{OX} = \frac{\pi R^4}{\cdot}$

$$\therefore \qquad (k_{OX})^2 = \frac{\pi R^4}{4 \times \pi R^2} = \frac{R^2}{4}$$

$$k_{OX} = \frac{R}{2} = \underline{\cdot 5R.}$$

To find the swing radius about a tangent—

$$(\text{distance})^2 \text{ (OX to YT)} = R^2$$

$$\therefore \qquad I_{YT} = I_{OX} + AR^2$$

$$= A\frac{R^2}{4} + AR^2 = A\frac{5}{4}R^2$$

hence $$k^2_{YT} = \frac{5}{4}R^2$$

or $$k_{YT} = \underline{1{\cdot}12R.}$$

Example 41.—To find the moment of inertia of a right circular cylinder of length h and radius R, about various axes.

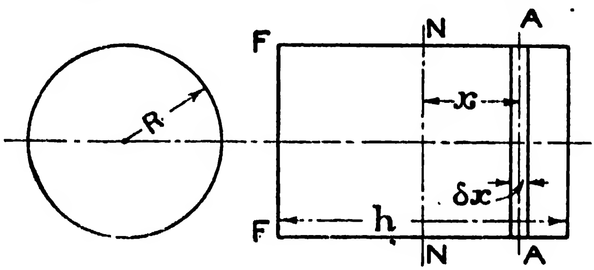


FIG. 85.

(*a*) About the axis of the cylinder.

(*b*) About an axis through the C. of G. perpendicular to the axis of the cylinder.

(*c*) About an axis parallel to that in (*b*), but through one end.

(*a*) The 2nd moment about the axis of the thin cylinder of length δx (Fig. 85)—

$$= \text{mass} \times \frac{R^2}{2} \quad \text{from } \textit{Example } 40$$

$$= \rho\pi R^2 \delta x \times \frac{R^2}{2} \quad \rho \text{ being the density of the material.}$$

Hence the total 2nd moment about the axis—

$$= \int_0^h \rho\pi R^2 \frac{R^2}{2}\, dx = \frac{\rho\pi R^4}{2} h$$

$$= \rho\pi R^2 h \times \frac{R^2}{2}$$

$$= \underline{m \frac{R^2}{2}}$$

where m = the mass of the cylinder.

(b) The 2nd moment of the strip about AA, which is parallel to NN—

$$= \text{mass} \times \frac{R^2}{4} \quad \text{(see } Example\ 40)$$

$$= \rho\pi R^2 \delta x \times \frac{R^2}{4}.$$

Hence I_{NN} of the strip $= I_{AA}$ of the strip$+$(its mass $\times x^2$) since AA is an axis through the C. of G. of the strip.

Thus I_{NN} of the strip $= \dfrac{\rho\pi R^4}{4}\delta x + \rho\pi R^2 x^2 \delta x$

and thus the total $I_{NN} = \displaystyle\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\rho\pi R^4}{4} dx + \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\pi R^2 x^2 dx$

$$= \frac{\rho\pi R^4}{4}\left(\frac{h}{2}+\frac{h}{2}\right) + \rho\pi R^2\left(\frac{h^3}{24}+\frac{h^3}{24}\right)$$

$$= \rho\pi R^2 h\left(\frac{R^2}{4}+\frac{h^2}{12}\right)$$

$$= m\left(\frac{R^2}{4}+\frac{h^2}{12}\right).$$

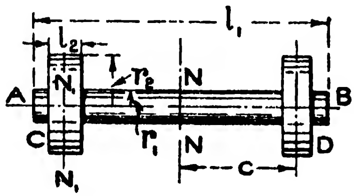


FIG. 86.

(c) To find I_{FF}, FF being parallel to AA and NN.

The distance between FF and NN $= \dfrac{h}{2}$.

Also it has just been proved that $I_{NN} = m\left(\dfrac{R^2}{4}+\dfrac{h^2}{12}\right)$.

Hence
$$I_{FF} = m\left(\frac{R^2}{4}+\frac{h^2}{12}\right) + m\left(\frac{h}{2}\right)^2$$

$$= m\left(\frac{R^2}{4}+\frac{h^2}{3}\right).$$

Example 42.—Find an expression for the moment of inertia of a large pulley wheel of outside radius R and thickness of rim t. Neglect the arms or spokes of the wheel.

Let r = inside radius of wheel, *i. e.*, $r = R - t$.

Then, using the result of *Example* 40, p. 246, we know that the moment of inertia of the wheel as solid $= \dfrac{\pi R^4}{2} \times \rho b$; from this must

be subtracted the moment of inertia of a disc of radius r, viz., $\frac{\pi r^4}{2}\times\rho b$ (ρ being the density of the material and b the breadth of the rim along the face).

Hence $$I_O = \frac{\pi R^4}{2}\rho b - \frac{\pi r^4}{2}\rho b = \frac{\pi\rho b}{2}(R^4-r^4)$$
$$= \frac{\pi\rho b}{2}(R^2+r^2)(R^2-r^2)$$
$$= \pi\rho b(R^2-r^2)\times\frac{R^2+r^2}{2}$$
$$= M\left(\frac{R^2+r^2}{2}\right) \quad . \quad . \quad . \quad . \quad (1)$$

where M is the mass of the wheel.

Writing $R-t$ for r, $I_O = \frac{M}{2}(R^2+R^2+t^2-2Rt) = \frac{M}{2}(2R^2-2Rt+t^2)$.

From (1) it will be seen that in order to get I_O as large as possible, R and r must be very nearly equal, *i. e.*, t must be very small compared with R. Thus for an approximation t^2 may be neglected in the expression for I_O, so that $I_O = \frac{M}{2}\times 2R(R-t) = MR(R-t)$.

Referring once again to (1), $I_O = M\left(\frac{R^2+r^2}{2}\right)$, *i. e.*, $Mk_O^2 = M\left(\frac{R^2+r^2}{2}\right)$ or $k_O^2 = \frac{R^2+r^2}{2}$ and $k_O = \cdot 707\sqrt{R^2+r^2}$. As an approximation for this the rule $k_O = \frac{1}{2}(R+r)$ is often used; k_O being thus taken as the average radius.

Moment of Inertia of Compound Vibrators.—To find the modulus of rigidity of a sample of wire by the method of torsional oscillations, various forms of vibrators may be used. In the calculations which follow the experiments, the moment of inertia of the vibrator occurs, so that it is necessary to understand how to obtain this. To illustrate by an example of one form of compound vibrator, suppose that the I about an axis through the C. of G. of the one shown in Fig. 86 is required.

Let m_1 = mass of AB, r_1 be its radius and l_1 its length
m_2 = mass of C and also of D, r_2 be its radius and l_2 its length.

Then I_{NN} of AB $= m_1\left(\frac{r_1^2}{4}+\frac{l_1^2}{12}\right)$ (from *Example* 41, p. 248)

and $I_{N_1N_1}$ of C $= m_2\left(\frac{r_2^2}{4}+\frac{r_1^2}{4}+\frac{l_2^2}{12}\right)$ $\left\{\begin{matrix}\text{for the inner radius}\\ = r_1\end{matrix}\right\}$

$$\therefore \qquad I_{NN} \text{ of } C = m_2\left(\frac{r_2^2}{4}+\frac{r_1^2}{4}+\frac{l_2^2}{12}\right)+m_2c^2 \qquad \left(\begin{array}{c}\text{by the parallel}\\ \text{axis theorem.}\end{array}\right)$$

This is also the I_{NN} of D.

$$\therefore \qquad \text{total } I_{NN} = m_1\left(\frac{r_1^2}{4}+\frac{l_1^2}{12}\right)+2m_2\left(\frac{r_2^2}{4}+\frac{r_1^2}{4}+\frac{l_2^2}{12}+c^2\right).$$

Maxwell's needle is a very convenient form of compound vibrator, and is utilised to determine the modulus of rigidity of the sample of wire by which it is supported. It consists essentially of a tube along which weights may be moved from one position to another the movement being a definite amount.

Referring to Fig. 87—

m_1 = the mass of each of the movable weights
m_2 = the mass of each of the fixed weights.

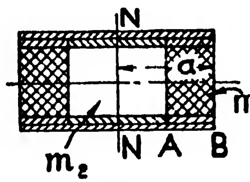


FIG. 87.

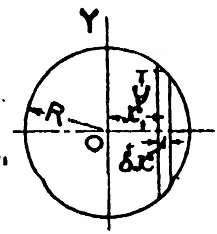


FIG. 88.

Then the time of torsional oscillations is measured when the movable weights are placed as shown, and again when they are moved to the centre; and it can be proved that the modulus of rigidity depends upon the difference between the moments of inertia under the two sets of conditions.

Thus, since a mass m_1 is shifted from the position AB to the position NA, the only difference in the moments of inertia is that due to the changing of the C. of G. of a mass (m_1-m_2) from a distance $\frac{3}{4}a$ from the axis of oscillation to $\frac{1}{4}a$; for I_{NN} of m_2 is unaltered.

$$\left.\begin{array}{l}\text{Hence the change of I}\\ \text{considering one mass only}\end{array}\right\} = I_1-I_2 = (m_1-m_2)\left(\frac{9}{16}a^2-\frac{1}{16}a^2\right)$$

and thus for the two masses $\quad I_1-I_2 = 2(m_1-m_2)\left(\frac{1}{2}a^2\right)$

$$= (m_1-m_2)a^2.$$

Example 43.—Find the moment of inertia of a sphere of radius R about its diameter

Consider the thin disc (Fig. 88) of radius y, and thickness δx.

I of the strip about a diameter parallel to OY—

$$= \frac{\pi}{4} y^4 \rho \delta x \quad \text{(cf. \textit{Example} 40, p. 246).}$$

Hence I of the strip about OY (distant x from the diameter considered)

$$= \frac{\pi\rho}{4} y^4 \delta x + (\pi\rho y^2 \delta x \times x^2)$$

Now $\quad y^2 = R^2 - x^2.$

Thus $\quad I_{OY}$ of disc $= \pi\rho\left\{\dfrac{R^4 + x^4 - 2R^2x^2 + 4R^2x^2 - 4x^4}{4}\right\}\delta x$

and hence $\quad I_{OY}$ of sphere $= \dfrac{\pi\rho}{4}\displaystyle\int_{-R}^{R}(R^4 - 3x^4 + 2R^2x^2)dx$

$$= \frac{2\pi\rho}{4}\int_0^R (R^4 - 3x^4 + 2R^2x^2)dx$$

$$= \frac{2\pi\rho}{4}\left[R^4x - \frac{3x^5}{5} + \frac{2R^2x^3}{3}\right]_0^R$$

$$= \frac{2\pi\rho}{4} \times \frac{16R^5}{15} = \frac{8}{15}\rho\pi R^5$$

$$= \frac{4}{3}\rho\pi R^3 \times \frac{2}{5}R^2$$

$$= \underline{m \times \frac{2}{5}R^2} \left(\begin{array}{c} m \text{ being the mass} \\ \text{of the sphere.}\end{array}\right)$$

and $\quad k^2_{OY} = \underline{\dfrac{2}{5}R^2}.$

Determination of 1st and 2nd Moments of Sections by means of a Graphic Construction and the Use of a Planimeter.

The graphic construction now to be described is extremely simple to understand, and has the additional merit of being utilised to give 3rd, 4th and higher moments if desired.

It being required to find the 1st and 2nd moments about MM of the rail section shown in Fig. 89, and also the position of the neutral axis, the procedure is as follows :—

Construction.—Divide the half-area into a number of strips by means of horizontal lines; the half-area only being treated, since the section is symmetrical.

At a convenient distance h from MM draw M_1M_1 parallel to MM. From P, the end of one of the horizontals, draw PR perpendicular to MM, and from P^1, the other end of the same horizontal, drop P^1R^1 perpendicular to M_1M_1; join R^1R and note Q, its point of intersection with P^1P. Repeat the process for all the other horizontals (of which only three are shown in the diagram) and

join up all the points like Q, thus obtaining the curve CQLS, which is termed the 1st moment curve.

To obtain the 2nd moment curve treat the area CPKXSLQ in the same way as the original area was treated, *i. e.*, drop QR″ perpendicular to M_1M_1 and join RR″; join up all points like Q^1 and the 2nd moment curve is obtained.

Calculation.—Find by the planimeter the areas of the original

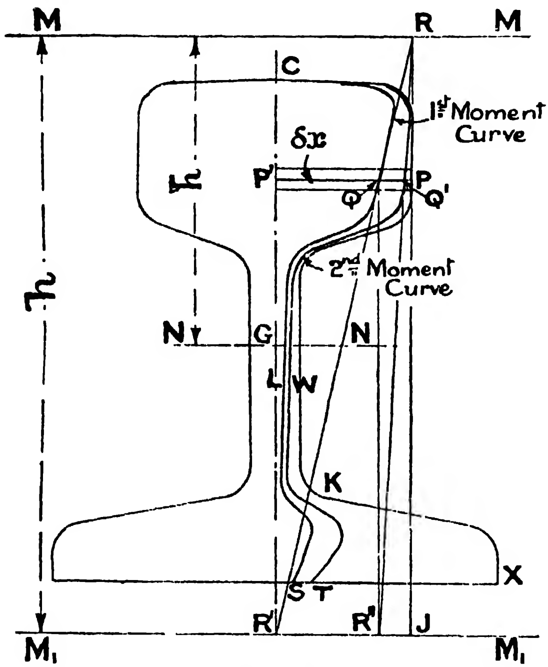


FIG. 89.—Moments of Sections by Graphic Construction.

half-section, CPKXSLQ and CPKXTWQ^1; call these A_0, A_1 and A_2 respectively.

Then 1st moment of the section about MM $= 2 \times hA_1$

(for A_1 is for the half-section only).

Distance of the centroid of the section from MM $= \dfrac{hA_1}{A_0}$

2nd moment of section about MM $= 2h^2A_2$

i. e., (swing radius $_{MM})^2 = \dfrac{h^2A_2}{A_0}$

and by the theorem of parallel axes, I can be found about NN.

In this case the actual results are as follows:—

$h = 3$ ins. $A_0 = 1{\cdot}11$ sq. ins. $A_1 = {\cdot}573$ sq. in. $A_2 = {\cdot}39$ sq. in.

Hence $$\bar{h} = \frac{3 \times {\cdot}573}{1{\cdot}11} = \underline{1{\cdot}55 \text{ ins.}}$$

1st moment of section about MM $= 2 \times 3 \times {\cdot}573 = \underline{3{\cdot}44 \text{ ins.}^3}$.

2nd moment of section about MM $= 2 \times 3^2 \times {\cdot}39 = \underline{7{\cdot}02 \text{ ins.}^4}$.

$$\text{Swing radius about MM} = \sqrt{\frac{7{\cdot}02}{2{\cdot}22}} = \underline{1{\cdot}78 \text{ ins.}}$$

N.B.—To distinguish which area is to be read off by the planimeter the following rule should be observed: Read the area between the 1st or 2nd moment curve, as the case may be, and the side of the original contour from which we dropped perpendiculars on the line about which we required moments.

Proof.—Consider P^1P as the centre line of a thin strip (such as the one indicated). Then the area of the strip $= P^1P \times \delta x$, and 1st moment about MM $= P^1P \times \delta x \times RP$.

From the similar triangles RPQ and RJR^1

$$\frac{RP}{QP} = \frac{RJ}{JR^1} = \frac{h}{PP^1}$$

whence $$RP \times PP^1 = h \times QP$$

and $$RP \times PP^1 \times \delta x = h \times QP \times \delta x$$

i. e., 1st moment of the strip about MM—

$= h \times$ the area of which QP is the centre line.

Then, by summing—

Total 1st moment of the half-area about MM—

$= h \times$ the area between 1st moment curve and right-hand boundary of section

$= hA_1$.

Again, the 2nd moment of the strip about MM $=$ area $\times$ (distance)^2

$= PP^1 \times \delta x \times (RP)^2$

and $$\frac{RP}{RJ} = \frac{PQ^1}{JR''} = \frac{PQ^1}{PQ}$$

$$\therefore \quad \frac{RP}{h} = \frac{PQ^1}{PQ} \quad i.\,e.,\ RP = \frac{hPQ^1}{PQ}.$$

Hence the 2nd moment of the strip about MM—

$= P^1P \times RP \times RP \times \delta x$

$= h \times QP \times h \times \frac{PQ^1}{PQ}\,\delta x = h^2 \times PQ^1 \times \delta x$

$= h^2 \times$ area of which PQ^1 is the centre line.

And the total 2nd moment of the half-area about MM—

$= h^2 \times$ area between the 2nd moment curve and the right-hand boundary of the section.

$= h^2 \times A_2$.

Exercise 20.—On Moment of Inertia.

1. Find the swing radius about the lighter end of a rectangular rod of uniform section and breadth and length l, for which the density is proportional to the square root of the distance from that end.

2. The swing radius of a connecting rod about its centre of suspension was found to be 35·8 ins., and the distance of the C. of G. from the point of suspension was 31·43 ins. Find the swing radius about the neutral axis.

If the connecting rod weighed 86·5 lbs., find its moment of inertia about the neutral axis.

FIG. 90.

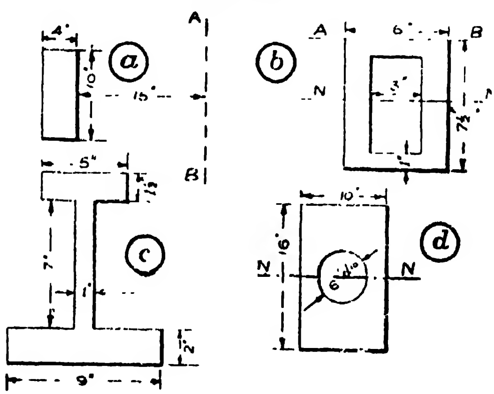


FIG. 91.

3. A circular disc, 7″ diameter, has a circular hole through it, of diameter 3″, the centre of the hole being ½″ distant from the centre of the disc. Find the swing radius of the disc about an axis through its centre of gravity, perpendicular to the face of the disc.

4. Find the moment of inertia of a rectangle (5″ by 3″) about a diagonal as axis.

5. Find the swing radius of a triangular plate (of height h)—

(a) When swinging about its base.

(b) When swinging about an axis through the vertex, parallel to the base.

6. By dividing into strips, by lines parallel to AB, find the moment of inertia, about AB, and also the swing radius, of the section shown at (a), Fig. 75, p. 234.

7. Find the radius of gyration about the axis of rotation, of the

rim of a flywheel, of outside diameter 5′ 2″, the radial thickness of the rim being 4″.

Find the moment of inertia about the neutral axis of the sections in Nos. 8, 9 and 10.

8. Channel Section, (*b*), Fig. 75.

9. Unequal Angle, (*c*), Fig. 75.

10. Tee Section, (*d*), Fig. 75.

11. Find the swing radius, about the axis, of a paraboloid, the diameter of the bounding plane, which is perpendicular to the axis, being d.

12. The *flexural rigidity* of a beam is measured by the product of the Young's Modulus E for the material into the moment of inertia of the section. Compare the flexural rigidity of a beam of square section with that of one of the same material but of circular section, the span and weight of the two beams being alike.

13. A cylinder 6″ long and of $1\frac{1}{2}$″ diameter is suspended horizontally by means of a long wire attached to a hook, and the wire is then twisted to give an oscillatory movement to the cylinder. Find the moment of inertia of the cylinder about the hook.

14. Determine the moment of inertia and also the swing radius about AB of the rectangular section shown at (*a*), Fig. 90.

15. Calculate the moment of inertia and also the swing radius of the box section shown at (*b*), Fig. 90, both about NN and about AB.

16. Find the position of the neutral axis of the section shown at (*c*), Fig. 91, and then calculate the moment of inertia and also the swing radius about this axis.

17. Determine the swing radius of the section shown at (*d*), Fig. 91, about the axis NN.

18. The moment of inertia of the pair of driving wheels of a locomotive connected by a crank axle was found by calculation to be 34133 lbs. ft.^{2}. If the total weight of the two wheels and the axle was 8473 lbs., and the diameter of the driving wheels was 6 ft. 1 in., find the swing radius of the wheel and also the ratio $\frac{k^2}{r^2}$, where r is the radius of the wheel.

19. Find the swing radius about the axis of a right circular cone of uniform density, the radius of the base being 5 ins.

20. Employing the method explained on p. 251, determine (*a*) the 1st moment about AB, (*b*) the 2nd moment about AB, (*c*) the distance of the centroid from AB, and (*d*) the swing radius about AB, of the area shown at (*b*) Fig. 76, p. 236.

21. A steel wire, ·15 in. in diameter, hangs vertically; its upper end is clamped, and its lower end is secured to the centre of a horizontal disc of steel, which is 6 in. in diameter and $\frac{3}{8}$ in. thick. If the length of the wire is 3 ft., and if C, the modulus of transverse elasticity of the steel, has the value 12,540,000 lbs. per sq. in., find the time of a torsional oscillation of the wire, from the formula—

$$t^2 = 402{\cdot}5\,\frac{\mathrm{I}l}{\mathrm{C}d^4}$$

where I = moment of inertia of the disc about the axis of suspension in lbs. ins.^{2}, l = length of wire in feet, d = diameter of wire in inches.

22. An anchor ring is generated by the revolution of a circle of radius r about an axis distant R from the centre of the circle. Find the moment of inertia of the ring about this axis. (*Hint.*—Commence with the polar moment, *i.e.*, the moment about the given axis, of an annulus made by a section at right angles to this axis, finding an expression for the inner and outer radii of the annulus in terms of the distance from the central annulus, and then sum up.)

23. Find the swing radius about the major axis of the ellipse whose equation is—

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

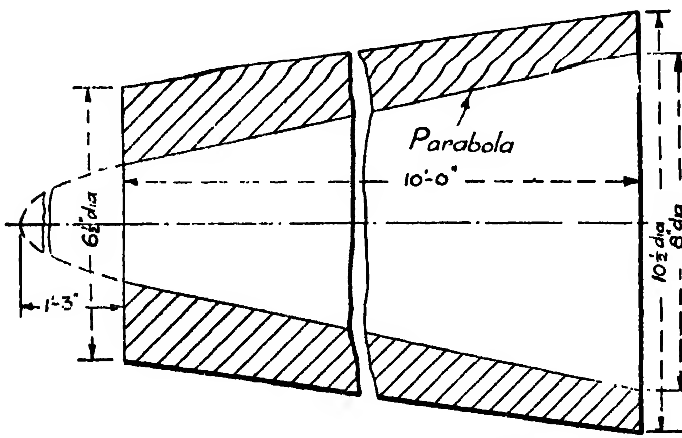


FIG. 91A.—Weight of a Centrifugal Casting (see p. 210).

24. Find the moment of inertia about the axis AA of the wheel and axle shown in Fig. 91B. The total mass is 9·1 lbs.

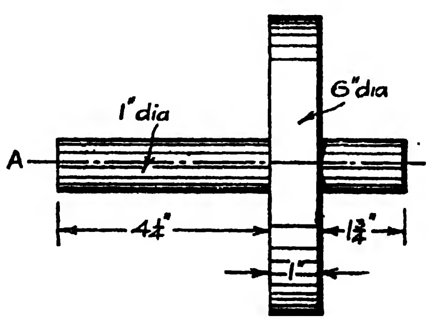


FIG. 91B.

CHAPTER VIII

POLAR CO-ORDINATES

Polar Co-ordinates.—A point on a plane may be fixed by its distances from two fixed axes, or by its distance along a line which makes a definite angle with some fixed axis. In the former case we are concerned with rectangular co-ordinates and the point is written as the point (x, y); whilst in the latter case the co-ordinates are *polar* and the point is denoted by (r, θ), r being the length along the ray inclined at an angle θ to the fixed axis.

It is really immaterial as to what line is taken as the fixed axis: in many cases the horizontal axis is taken, but in order to agree with the convention adopted for the measurement of angles (see Part I, Chapter VI) we shall here consider the N. and S. line, *i. e.*, a vertical line, as the starting axis and regard all angles as positive when measured in a right-handed direction from that axis. A point is next fixed on that line from which all the rays or radii vectors originate, and this point is spoken of as the *pole* for the system: thus the reason for the term *polar* is seen.

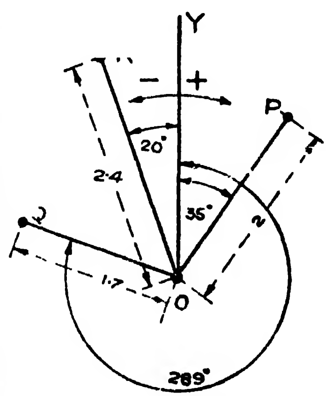


FIG. 92.

To illustrate this method of plotting, let us refer to Fig. 92. Taking OY as the starting axis and O as the pole, the point (2, 35°) is obtained by drawing a line making 35° with OY and then stepping off a distance OP along it to represent 2 units, *i. e.*, $r=2$ and $\theta=35°$. In like manner Q is the point (1·7, 289°); whilst R is the point (2·4, −20°).

One advantage of this method of plotting is that it is not necessary to classify into quadrants and to remember the arrange-

ment of the algebraic signs; all lengths measured outwards from the pole being reckoned as positive.

Example 1.—The following table gives the candle power of an arc lamp for various positions below the lamp: plot the polar diagram.

Angle below horizontal . .	0	10°	20°	30°	40°	50°	60°	70°	80°	90°
Candle power	1000	1470	1800	1720	1200	960	800	720	600	480

In reality we have to plot a number of polar co-ordinates, the lengths representing the values of the candle power; but since the horizontal axis is specified, we shall take that as the main axis. Draw rays making 10°, 20°, 30°, etc. (Fig. 93), with the horizontal axis, and along these lines set off distances to represent the respective candle powers, always measuring outwards from the centre. Join the ends of the rays and the polar diagram is completed.

The Archimedean spiral and the logarithmic or equiangular spiral, important in connection with the forms of cams and gear wheels respectively, may be easily plotted from their polar equations.

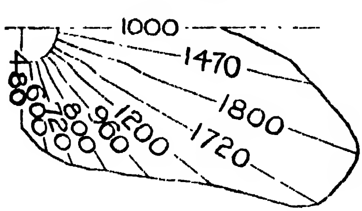


FIG. 93.—Candle Power of Arc Lamp.

Thus the equation to the Archimedean spiral is $r=a\theta$, and the equation to the equiangular spiral is $r=ae^{b\theta}$; indicating that in the former case the rays, for equal angular intervals, are the consecutive terms in an arithmetic progression, whilst in the latter case the rays are in geometric progression.

To illustrate the forms of these curves by taking numerical examples:—

Example 2.—Plot the Archimedean spiral $r=\cdot573\theta$, showing one convolution.

In the equation θ must be in radians, but to simplify the plotting we can transform the equation so that values of a (in degrees) may replace θ (radians).

Thus— $$r = \cdot573\theta = \frac{\cdot573a}{57\cdot3} \text{ (degrees)} = \cdot01a.$$

Then the table for the plotting reads :—

a	0	30	60	90	120	150	180	210	240	270	300	330	360
r	0	·3	·6	·9	1·2	1·5	1·8	2·1	2·4	2·7	3·0	3·3	3·6

and the plotting is shown in Fig. 94.

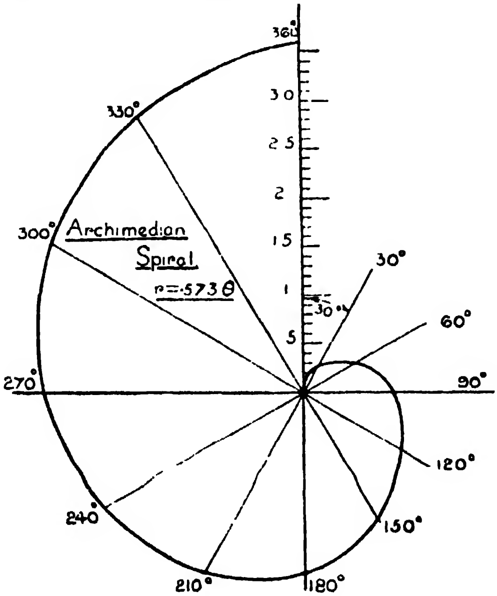


FIG. 94.

Example 3.—Plot one convolution of the equiangular spiral

$$r = \cdot 5e^{\cdot 25\theta}.$$

$$r = \cdot 5e^{\cdot 25\theta} = \cdot 5e^{\frac{\cdot 25a}{57\cdot 3}} = \cdot 5e^{\cdot 00436a}.$$

In the log form $\quad \log r = \log \cdot 5 + \cdot 00436a \log e$

$$= \bar{1}\cdot 6990 + (\cdot 00436 \times \cdot 4343 \times a)$$

$$= \bar{1}\cdot 6990 + \cdot 001894a$$

and thus the table of values reads :—

.	0	30	60	90	120	150	180	210	240	270	300	330	360
$\cdot 001894a$	0	·0568	·1136	·1705	·2273	·2841	·3409	·3977	·4546	·5114	·5682	·625	·6818
$\log r$.	$\bar{1}$·6990	$\bar{1}$·7558	$\bar{1}$·8126	$\bar{1}$·8695	$\bar{1}$·9263	$\bar{1}$·9831	·0399	·0967	·1536	·2104	·2672	·324	·3808
r . . .	·5	·5699	·0495	·7407	·8439	·9618	1·096	1·249	1·424	1·624	1·85	2·109	2·403

The curve is drawn in Fig. 95.

It will be seen that the ratio of

$$\frac{\text{second ray}}{\text{first ray}} = \frac{\cdot 5699}{\cdot 5} = 1\cdot 14$$

$$\frac{\text{third ray}}{\text{second ray}} = \frac{\cdot 6495}{\cdot 5699} = 1\cdot 14$$

so that this spiral might alternatively have been defined as one for which the rays at equiangular intervals of 30° form a geometric progression in which the common ratio is 1·14, the first ray being ·5".

Comparing the given equation $r = \cdot 5e^{\cdot 25\theta}$ with that connecting the tensions at the ends of a belt passing round a pulley, viz., $T = te^{\mu\theta}$, we observe that the forms are identical, or in other words the equiangular spiral might be used to demonstrate the growth of the tension as the belt continuously embraces more of the pulley.

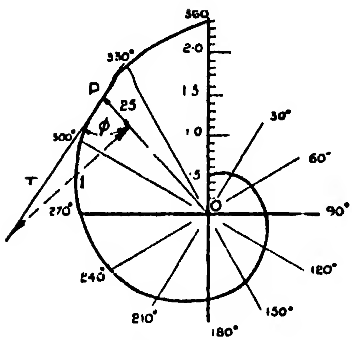


FIG. 95.

Selecting any point P on the spiral, and drawing the tangent PT there and also the ray OP which makes an angle ϕ with the tangent, it is found that $\cot \phi = \cdot 25 =$ coefficient of θ in the original equation. This relation would hold wherever the point P was taken on the spiral, so that the angle between the ray and the curve is constant: and thus the spiral is called "equiangular."

If $\cot \phi = 1$, $\phi = 45°$ and $r = ae^{\theta}$, or taking $a = 1$, $r = e^{\theta}$ and $\log_e r = \theta$. Thus a spiral could be constructed in which the angles (in radians) would be the values of the logs of the rays: this spiral, however, is extremely tedious to draw, and its value consists merely in its geometric demonstration of the relationship between the natural logarithms and their numbers.

Connection between Rectangular and Polar Co-ordinates.—Let P be a given point, with rectangular co-ordinates x and y and with polar co-ordinates r and θ.

Then referring to Fig. 96—

$$\frac{\text{ON}}{\text{OP}} = \frac{y}{r} = \cos \theta$$

so that $$y = r \cos \theta$$

and $$\frac{\mathrm{OM}}{\mathrm{OP}} = \frac{x}{r} = \sin\theta$$

so that $$x = r\sin\theta$$

and also $$\frac{x}{y} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta.$$

Use of Polar Co-ordinates for the Determination of Areas.—Polar co-ordinates may be usefully employed to find areas of certain figures.

It is stated in the previous work on mensuration that—

$$\text{Area of sector of circle} = \tfrac{1}{2}r^2\theta$$

where θ = angle of the sector in radians.

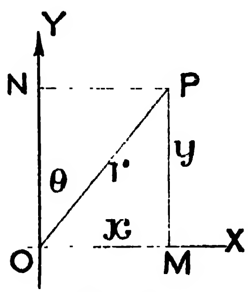


FIG. 96.

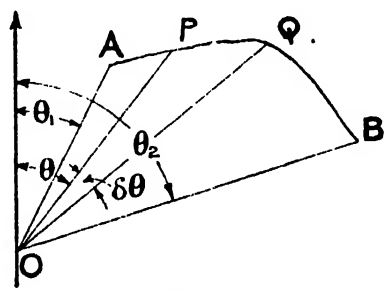


FIG. 97.

Let P and Q (Fig. 97) be the two points (r, θ) and $(r+\delta r, \theta+\delta\theta)$ and close to one another.

Then, since r and $r+\delta r$ differ very slightly

$$\text{Area POQ} = \tfrac{1}{2}r^2\delta\theta$$

and the total area AOB $= \sum_{\theta_1}^{\theta_2}\tfrac{1}{2}r^2\delta\theta$, approximately

or $\int_{\theta_1}^{\theta_2}\tfrac{1}{2}r^2 d\theta$ exactly.

For the evaluation of this integral the working may be either graphic or algebraic, according to the manner in which the relation between r and θ is stated.

As a simple illustration we may take the case of a circle of radius a. The area of the circle was found at an earlier stage (see p. 225) by evaluating $\int y dx$, *i. e.*, by expressing the integral in terms of the rectangular co-ordinates. To evaluate the integral, however, it was found necessary to make the substitution $x = a\sin\theta$, the change thus being from rectangular to polar co-ordinates.

Evidently the rotating ray is constant in length and equal to a, the radius of the circle, and the limits to θ are 0 and 2π, if the full area is required; hence—

$$\text{Area of circle} = \int_0^{2\pi} \tfrac{1}{2}a^2 d\theta = \tfrac{1}{2}a^2 \int_0^{2\pi} d\theta = \tfrac{1}{2}a^2 \cdot 2\pi$$
$$= \pi a^2.$$

Example 4.—Find the area of the cardioid given by the equation $r = a(1 + \cos\theta)$, θ ranging from 0 to 2π.

In this case r is of variable length, but there is a definite connection between r and θ, so that the integration is algebraic.

$$r = a(1+\cos\theta) \text{ and } r^2 = a^2(1+\cos\theta)^2$$
$$= a^2(1+\cos^2\theta + 2\cos\theta)$$
$$= a^2(1+\tfrac{1}{2}\cos 2\theta + \tfrac{1}{2} + 2\cos\theta)$$
$$= a^2(1\cdot 5 + \tfrac{1}{2}\cos 2\theta + 2\cos\theta).$$

Hence

$$\text{area} = \int_0^{2\pi} \tfrac{1}{2}r^2\, d\theta$$
$$= \frac{a^2}{2}\int_0^{2\pi} (1\cdot 5 + \tfrac{1}{2}\cos 2\theta + 2\cos\theta)\, d\theta$$
$$= \frac{a^2}{2}\left(1\cdot 5\theta + \tfrac{1}{4}\sin 2\theta + 2\sin\theta\right)_0^{2\pi}$$
$$= \frac{a^2}{2}(1\cdot 5 \times 2\pi) = \underline{\frac{3\pi a^2}{2}}.$$

The Rousseau Diagram.—The use of the Rousseau diagram simplifies the determination of the *mean spherical candle power* of a lamp.

The candle power of the lamp varies according to the direction in which the illumination is directed (cf. Fig. 93); in the case there discussed, however, we considered the illumination in one plane only. If we imagine the polar curve to revolve round the vertical axis we see that a surface is obtained by means of which the illumination in any direction can be measured. The mean of all these candle powers is spoken of as the mean spherical candle power of the lamp. If the arc is placed at the centre of a spherical enclosure, of radius R, then, if I_M is the mean spherical candle power (M.S.C.P.) of the lamp, the total illumination is expressed by $4\pi R^2 I_M$: this total might be arrived at, however, by summing the products of the candle power in any direction into the area of the zone over which this intensity is spread; and putting this statement into the form of an equation,

$$4\pi R^2 I_M = \Sigma IA$$

where I is the intensity on a zone of surface area A.

To find the M.S.C.P. proceed as follows:—Suppose that the lamp is at O (Fig. 98). With centre O and any convenient radius R describe a semicircle; also let the polar diagram be as shown (the curve OPQMC). The greatest candle power is that given by OC; draw a horizontal through N, the point in which the line OC meets the circumference of the semicircle, and make $ab = OC$. Through a and b draw verticals and through A and B draw horizontals, thus obtaining the rectangle DE; draw a number of rays, OP, OQ, OS, etc., and also horizontals through the points p, q, s, etc., marking along these lines distances equal to OP, OQ, OS, etc., working from DF as base. By joining up the points so obtained the curve FLbD is obtained, known as the Rousseau

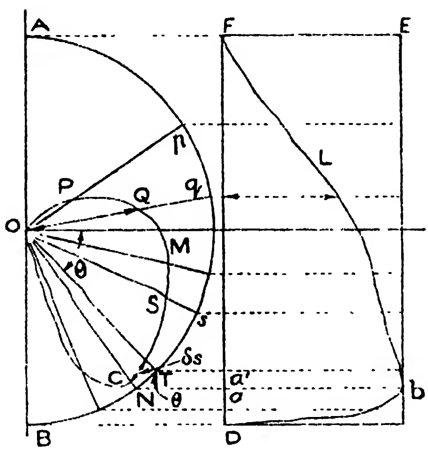


FIG. 98.

curve; then the mean height of this curve (which can readily be obtained by means of a planimeter) gives the M.S.C.P. of the lamp.

Proof of this Construction.—Let I_M = M.S.C.P. of the lamp

$$\text{then} \qquad I_M = \frac{\Sigma \text{ area of zone} \times \text{C.P.}}{4\pi R^2}.$$

Consider the zone generated by the revolution of TN ($= \delta s$) about AB; its area is of the form $2\pi y \delta s$ and the intensity of illumination is OC, say. The length y is the projection on the horizontal axis through O of either the line OT, the line ON, or the line midway between these (for these differ in length but slightly if δs is taken as very small), *i. e.*, $y = \text{OT} \cos\theta$

$$\text{or} \qquad y = R\cos\theta.$$

Hence, for this zone, the illumination

$$= \text{candle power} \times \text{area}$$
$$= OC \times 2\pi R \cos\theta\delta s$$
$$= ab \times 2\pi R \times aa' \left\{ \text{for } \frac{aa'}{\delta s} = \cos\theta \right\}.$$

Hence the total illumination

$$= 2\pi R \Sigma ab \times aa'$$
$$= 2\pi R \times \text{area under the curve FL}b\text{D}$$

and thus

$$I_M = \frac{\text{Total illumination}}{4\pi R^2} = \frac{2\pi R \times \text{area under the curve FL}b\text{D}}{4\pi R^2}$$
$$= \frac{\text{area under the curve FL}b\text{D}}{2R}$$
$$= \text{mean height of the curve FL}b\text{D}$$

since 2R is the base of the curve.

Fleming's Graphic Method for the Determination of R.M.S. Values.—The determination of R.M.S. values is of some importance to electrical engineers, and the subject, previously discussed in Chapter VII, is here treated from a different aspect. Instead of squaring the given values of the current and then extracting the square root of the mean of these squares, we may, by a simple graphic construction, obtain the mean of the squares very readily.

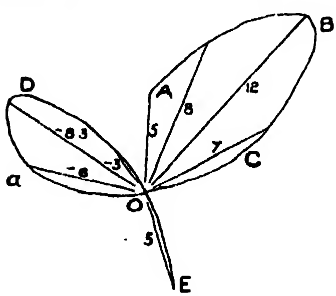


FIG. 99.

Let the values of an alternating current at various times be as in the table:—

time t . .	0	·001	·002	·003	·004	·005	·006	·007	·008
Current I	5	8	12	7	0	−6	−8·3	−3	5

then, to find the R.M.S. value of I we proceed as follows:—

Treat the given values as polar co-ordinates, taking t for the angles and I for the rays. Select some convenient scale for t, say $20° = ·001$ sec., and a scale for I, say $1'' = 4$ units, these being the scales chosen for the original drawing of which Fig. 99 is a copy to

about one-half scale; and set out a polar diagram as indicated, making $OA = +5$, $Oa = -6$, etc. Join the extremities of the rays, so obtaining, with the first and last rays, the closed figure ABCD*a*E. Measure the area of this figure by means of a planimeter—in this case the area was found to be 4·23 sq. ins.

Now the area of the figure $= \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int I^2 dt$, or

$$2 \times \text{area} = \int I^2 dt$$

so that if we divide twice the area by the range in t, the mean value of the squares is determined.

In this case the range of $t = 160° = 2{\cdot}79$ radians, and also $\int I^2 dt = 2 \times 16 \times 4{\cdot}23 = 135{\cdot}5$, for $1'' = 4$ units, and thus 1 sq. in. $= 16$ sq. units.

Then $$\text{M.S.} = \frac{135{\cdot}5}{2{\cdot}79} = 48{\cdot}65,$$

and hence $$\text{R.M.S.} = \sqrt{48{\cdot}65} = 6{\cdot}98.$$

The rule for the area of a figure, viz., $\frac{1}{2}\int r^2 d\theta$, may be usefully employed to find the height of the centroid of an area above a certain base.

Example 5.—Find the height above the base OX of the centroid of the irregular area OABX [(*a*) Fig. 100].

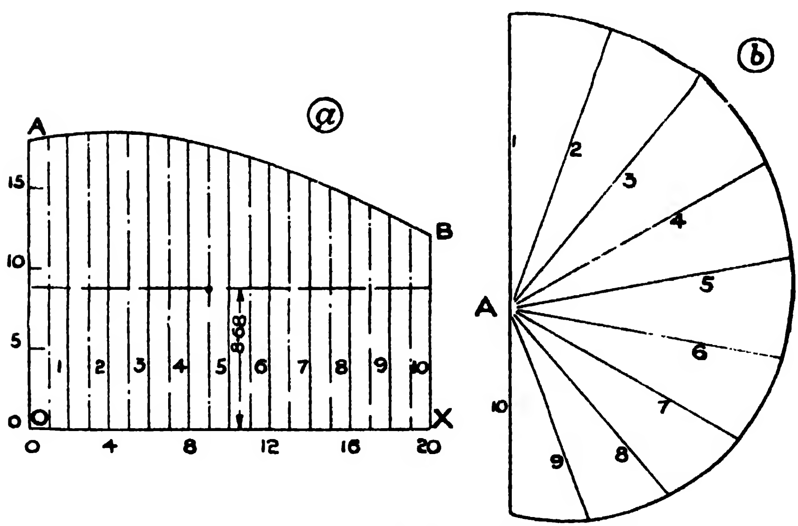


Fig. 100.—Centroid of Irregular Area by Polar Diagram.

To do this, first divide the base into a number of equal divisions and erect mid-ordinates in the usual way. Measure these mid-ordinates and set off lengths to represent them as radii vectors from A in (*b*) Fig. 100, the angles at which the rays are drawn representing the values of x, *i. e.*, the lengths of the divisions of the base. Join the ends of these rays and measure the area of the polar diagram thus obtained; divide this area by the area of the original figure and the result is the height $\bar{y}$ required.

For the area of the polar diagram $= \frac{1}{2}\int r^2\, d\theta = \frac{1}{2}\int y^2\, dx$, since rays represent values of y and the angles represent values of x. Also the area of the original figure $= \int y\, dx$, so that

$$\frac{\text{area of polar diagram}}{\text{area of original figure}} = \frac{\frac{1}{2}\int y^2\, dx}{\int y\, dx} = \bar{y} \text{ (cf. p. 214).}$$

For the particular case illustrated (the scales referring to the original drawing) :—

For (*a*) Fig. 100 $\quad 1'' = 5$ units vertically
$1'' = 4$ units horizontally

so that 1 sq. in. represents 20 units of area. The area was found by the planimeter to be 16·82 sq. ins., so that the actual area is 336·4 sq. units, *i. e.*, $\int y\, dx = 336{\cdot}4$.

For (*b*) Fig. 100 $\quad 1'' = 5$ units radially

and each angular interval $= 20^\circ$, so that the total range $= 180^\circ$ or 3·14 radians. Hence 3·14 radians represent 20 units, the length of the base in (*a*) Fig. 100,

or $$1 \text{ radian} = \frac{20}{3{\cdot}14} = 6{\cdot}36 \text{ units.}$$

Now the area is of the nature $r^2 \times \theta$, *i. e.*, $(\text{length})^2 \times \text{angle}$, hence 1 sq. in. of area $= 5^2 \times 6{\cdot}36$ or 159 units. Area of the polar figure (found by the use of the planimeter)

$$= 18{\cdot}34 \text{ sq. ins.} = 18{\cdot}34 \times 159 \text{ units} = 2920.$$

Hence $$\bar{y} = \frac{2920}{336{\cdot}4} = \underline{8{\cdot}68},$$

i. e., the centroid horizontal is found.

Theory of the Amsler Planimeter. The principle upon which the planimeter is based may be explained quite simply, in the following way.

In Fig. 101 let PP″ be a portion of the outline of the figure whose area is to be measured, and let the fixed centre of the instrument be at O. Then in the movement of the tracing point P from P to P″ along the curve, the tracing arm changes from the position AP to A′P″. This movement may be regarded as made up of two distinct parts: firstly, a sliding or translational movement from AP to A′P′, and next, a rotation round A′ as centre,

from A′P′ to A′P″. In the former of these movements the recording wheel moves from W to W′, but part of this movement only, viz., that perpendicular to the axis of the wheel, is actually recorded, so that the wheel records the distance p.

The area swept out by the tracing arm AP during the small change from P to P″ = APP′P″A′ = APP′A′ + P′P″A′

$$= (AP \times p) + (\tfrac{1}{2}(AP)^2\,\delta\theta).$$

Hence for the whole area,

$$\text{area swept out} = \Sigma AP \times p + \Sigma \tfrac{1}{2}(AP)^2\,\delta\theta$$
$$= AP\Sigma p + \tfrac{1}{2}AP^2\Sigma\delta\theta.$$

Now the net angular movement is zero, so that $\Sigma\delta\theta = 0$.

Hence area swept out $= AP\Sigma p$,

or if l = length of the tracing arm,

area $= l \times$ travel of wheel.

and hence the reading of the wheel

$$= \frac{\text{area of figure}}{l}$$

Thus the length of the tracing arm determines the scale to which the area is measured. Hence by suitable adjustment of this length of arm the area of a figure may be read in sq. ins. or sq. cms. as may be necessary. If the average height of the figure is required, the length of the tracing arm must be made exactly equal to the length of the figure. This is done by using the points LL (Part I, Fig. 301), and not troubling about the adjustment at A. The difference between the first and last readings gives, when multiplied or divided by a constant, the actual mean height of the figure. If the ordinary Amsler is used, then the mean height in inches is obtained by dividing the difference between the readings by 400; thus if the first reading was 7243 and the last 7967, the mean height would be the difference, viz., 724, divided by 400, *i.e.*, 1·81 ins.

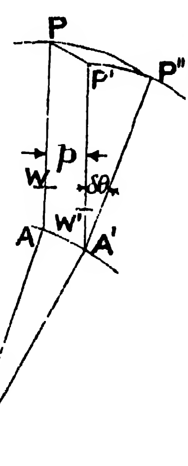


FIG. 101.—Theory of Amsler Planimeter.

The area of the figure = average height × length, but the area of the figure = length of tracing arm × wheel reading, hence if the length of the tracing arm = the length of the diagram, the wheel reading must be the average height of the diagram.

[It should be noticed that the area recorded by the instrument is really the difference between the areas swept out by the ends A and P of the arm AP, but as A moves along an arc of circle, coming back finally to its original position, no area is swept out.]

Exercises 21.—On Polar Co-ordinates.

1. Plot a polar curve of crank effort for the following case, the connecting rod being infinitely long.

θ	0	15	30	45	60	75	90	105	120	135	150	165	180
Crank Effort (lbs.) . .	0	2·4	3·9	5·1	6·4	7·1	7·3	7	6·1	4·9	3·3	1·5	0

2. As for Question 1, but taking the connecting rod = 5 cranks.

θ	0	15	30	45	60	75	90	105	120	135	150	165	172·5	180
C.E. (lbs.)	0	2·45	4·4	6·1	7·0	7·4	7·4	6·6	5·8	4·4	3·0	1·5	·6	0

3. An A.C. is given by $I = 7{\cdot}4 \sin 50\pi t$. Draw the polar curve to represent the variation in I and hence find its R.M.S. value.

4. What is the polar equation of a circle, the extremity of a diameter being taken as the centre from which the various rays are to be measured?

Of what curve (to Cartesian ordinates, *i. e.*, rectangular axes) is the circle the polar curve?

5. Plot the polar diagram for the arc lamp, from the table.

Angle (degrees) . . .	0	10	20	30	40	50	60	70	80	90° (vertical)
Candle Power	800	1200	1600	2050	2200	2200	2300	2500	2300	1800

6. Plot the Rousseau diagram for the arc lamp in Question 5 and from it calculate the M.S.C.P. of the lamp.

7. An A.C. has the following values at equal intervals of time: 3, 4, 4·5, 5·5, 8, 10, 6, 0, −3, −4, −4·5, −5·5, −8, −10, −6, 0. Find by Fleming's method (cf. p. 264) the R.M.S. value of this current.

8. Eiffel's experiments on the position of the centre of pressure for a flat plane moved through air at various inclinations gave the following results:—

Inclination to horizontal	0	5	10	15	30	45	60	75	90
Ratio $\frac{BC}{AB}$ (see Fig. 102)	·263	·291	·333	·375	·41	·415	·445	·47	·5

Plot a polar diagram to represent the variation of this ratio.

9. Draw the polar curve to represent the illuminating power of a U.S. standard searchlight from the following figures:—

Angle (degrees)	0 (vertical)	10	20	30	40	50	60	70	80	90
Candle Power.	3000	10000	20500	33000	41500	41500	43000	43000	30000	24000

10 (above horizontal)	20	30	40	50	60	70	80	90
9000	6000	5000	5000	2000	3000	1500	1500	1500

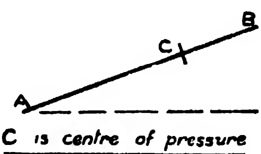


FIG. 102.

CHAPTER IX

SIMPLE DIFFERENTIAL EQUATIONS

Differential Equations.—An equation containing one or more derived functions is called a "differential equation."

Thus a very simple form of differential equation is

$$\frac{dy}{dx}=5$$

and $$4\frac{d^2y}{dx^2}+7\frac{dy}{dx}-5=0$$

and $$y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}-y=0$$

are more complex forms.

Differential equations are classified according to "order" or "degree"; the order being fixed by that of the highest differential coefficient occurring in it. Thus $\frac{dy}{dx}$ is a differential coefficient of the first order, $\frac{d^3y}{dx^3}$ is of the third order, and so on.

Hence $4{\cdot}2+y\frac{dy}{dx}=5{\cdot}34$ is an equation of the first order

and $8\frac{d^4y}{dx^4}+y=7{\cdot}16$ is an equation of the fourth order.

The "degree" of an equation is fixed by that of the highest derivative occurring when the equation is free from radicals and fractions.

Thus $\frac{d^2y}{dx^2}=c$ is of the second order and of the first degree

whilst $4\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{d^2y}{dx^2}\right)=7$ is of the second order and of the second degree.

Much has been written concerning the solution of the many types of differential equations, but it is only possible here to treat

the forms that are likely to arise in the derivation of the proofs of engineering formulæ; the plan being to discuss the solution according to the types of equation.

Type :— $\frac{dy}{dx}$ **given as a function of** x.—With the solution of such simple forms we have already become familiar, for the equations connecting the bending moment at various sections of a beam with the distances of those sections from some fixed point are of this character.

Thus taking the case of a simply supported beam carrying a load W at the centre

$$\frac{d^2y}{dx^2}=\frac{W}{EI}$$

$$\text{whence } \frac{dy}{dx}=\frac{Wx}{EI}+C$$

which is of the type under consideration.

Evidently this equation can be solved by integration throughout, attention being paid to the constants which are necessarily introduced. Expressing in algebraical symbols,

$$\text{If} \qquad \frac{dy}{dx}=f(x)$$

then by integrating throughout with regard to x

$$\int\frac{dy}{dx}dx=\int f(x)dx+C$$

$$\text{or} \qquad y=\int f(x)dx+C.$$

Example 1.—If $\frac{dy}{dx}=4x^2+7x-2$ and $y=5$ when $x=1$, find an expression for y in terms of x.

This equation is of the type with which we are now dealing, since

$$4x^2+7x-2=f(x)$$

$$\frac{dy}{dx}=4x^2+7x-2.$$

$$\text{Integrating} \qquad y=\frac{4x^3}{3}+\frac{7x^2}{2}-2x+C.$$

The value of C must now be found: thus $y=5$ when $x=1$

so that $\qquad 5=\frac{4}{3}+\frac{7}{2}-2+C$

or $\qquad C=2\cdot17.$

Hence $\qquad \underline{y=1\cdot33x^3+3\cdot5x^2-2x+2\cdot17.}$

Type :— $\dfrac{dy}{dx}$ **given as a function of** y. *i. e.*, $\dfrac{dy}{dx}=f(y)$.

This type of equation differs somewhat from the preceding in that a certain amount of transposition of terms has to be effected before the integration can be performed.

The equation may be written $\dfrac{dy}{f(y)} = dx$

the transposition being spoken of as "separating the variables," and thence by integration $\displaystyle\int\frac{dy}{f(y)} = \int dx + C = x + C.$

Example 2.—If $\dfrac{dy}{dx} = y^3$, find an expression for y.

Separating the variables $\qquad \dfrac{dy}{y^3} = dx.$

Integrating $\qquad \displaystyle\int\frac{dy}{y^3} = \int dx + C$

or $\qquad -\frac{1}{2}y^{-2} = x + C$

whence $\qquad x + \dfrac{\cdot 5}{y^2} + C = 0.$

The two following examples are really particular cases of the type discussed generally on p. 275, but they may also be included here as illustrations of this method of solution.

Example 3.—Solve the equation $\dfrac{dy}{dx} + ay = b$.

Here $\qquad \dfrac{dy}{dx} = b - ay$

or $\qquad \dfrac{dy}{b-ay} = dx$

so that $\qquad \displaystyle\int\frac{dy}{b-ay} = \int dx + C$

i. e., $\qquad -\dfrac{1}{a}\log_e(b-ay) = x + C$

or $\qquad \log_e(b-ay) = -(ax + aC)$

whence $\qquad e^{-ax-aC} = (b-ay).$

Now let $\quad A = e^{-aC}$: then $e^{-ax-aC} = e^{-ax} \times e^{-aC} = Ae^{-ax}$

and $\qquad Ae^{-ax} - b = -ay$

or $\qquad y = \dfrac{b}{a} - \dfrac{A}{a}e^{-ax}.$

Example 4.—Solve the equation $4\frac{dy}{dx} = 11+7y$.

Separating the variables

$$\int \frac{4dy}{11+7y} = \int dx + C$$

i. e., $$\tfrac{4}{7}\log_e(11+7y) = x + C$$

or $$\log_e(11+7y) = \tfrac{7}{4}x + \tfrac{7}{4}C$$

whence $$e^{\frac{7}{4}x+\frac{7}{4}C} = (11+7y)$$

and if A be written in place of $e^{\frac{7}{4}C}$

$$11+7y = Ae^{\frac{7}{4}x}$$

or $$\underline{y = \frac{A}{7}e^{\frac{7}{4}x} - \frac{11}{7}.}$$

Example 5.—The difference in the tensions at the ends of a belt subtending an angle of $d\theta$ at the centre of a pulley $= dT = T\mu d\theta$, where μ is the coefficient of friction between the belt and pulley. If the greatest and least tensions on the belt are T and t respectively, whilst the lap is θ, find an expression for the ratio $\frac{T}{t}$.

The equation $dT = T\mu d\theta$ is of the type dealt with in this section; to solve it we must separate the variables, thus:—

$$\frac{dT}{T} = \mu d\theta.$$

Integrate both sides of the equation, applying the limits t and T to T and 0 and θ for the angle.

Then $$\int_t^T \frac{dT}{T} = \mu\int_0^\theta d\theta.$$

$\therefore$ $$\Big(\log_e T\Big)_t^T = \mu\Big(\theta\Big)_0^\theta.$$

$\therefore$ $$\log_e T - \log_e t = \mu\theta.$$

But $$\log_e\frac{T}{t} = \log_e T - \log_e t.$$

Hence $$\log_e\frac{T}{t} = \mu\theta$$

or $$\underline{\frac{T}{t} = e^{\mu\theta}.}$$

A word further might be added about *Example* 3, or a modification of it.

Let $$\frac{dy}{dx} = ay.$$

If $a=1$, then $\frac{dy}{dx}=y$, *i. e.*, the rate of change of y with regard to x, for any value of x, is equal to the value of y for that particular value of x. Now we have seen (Part I, p. 353) that this is the case only when $y=e^x$.

If a has some value other than 1, y must still be some power of e, for the rate of change of y is proportional to y; actually, if $y=e^{ax}$, $\frac{dy}{dx}=ae^{ax}=ay$, so that $y=e^{ax}$ would be one solution of the equation $\frac{dy}{dx}=ay$; but to make more general we should write the solution in the form $y=e^{ax+c}$ or $y=Ae^{ax}$, whichever form is the more convenient. Whenever, therefore, one meets with a differential equation expressing the Compound Interest law (*i. e.*, when the rate of change is proportional to the variable quantity) one can write down the solution according to the method here indicated.

Example 6.—Find the equation to the curve whose sub-normal is constant and equal to $2a$.

$$\text{The sub-normal} = y\frac{dy}{dx}. \qquad \text{(See p. 43.)}$$

Thus
$$y\frac{dy}{dx}=2a,$$

or, separating the variables, $\int y\,dy=\int 2a\,dx$.

Hence
$$\frac{y^2}{2}=2ax+C \qquad \text{(Integrating)}$$

or
$$\underline{y^2=4ax+K.}$$

This is the equation of a parabola; if $y=0$ when $x=0$, then $K=0$ and $y^2=4ax$, *i. e.*, the vertex is at the origin.

Example 7.—Find an expression giving the relation between the height above the ground and the atmospheric pressure; assuming that the average temperature decrease is about 3·5° F. per 1000 feet rise, and the ground temperature is 50° F.

Let τ be the absolute temperature at a height h, then, from hypothesis

$$\tau=460+50-\frac{3\cdot5}{1000}h$$

$$=510-\cdot0035h \quad \ldots\ldots\ldots\ldots \quad (1)$$

Now we know that $pv=C\tau$ . . . . (2)

and also that if a small rise δh be considered, the diminution in the

pressure, viz., δp, is due to a layer of air δh feet high and 1 sq. ft. in section, and thus

$$\delta p = -\frac{\delta h}{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From equations (1) and (2)

$$pv = C(510 - \cdot 0035h)$$

and substituting for v its value from equation (3)

$$-\frac{p\delta h}{\delta p} = C(510 - \cdot 0035h)$$

or, in the limit

$$-\frac{dp}{pdh} = \frac{1}{C(510 - \cdot 0035h)}$$

Separating the variables

$$-\frac{dp}{p} = \frac{dh}{C(510 - \cdot 0035h)}$$

Integrating, the limits to p being p_0 and p, and those to h being o and h,

$$\left(-\log p\right)_{p_0}^{p} = \frac{1}{C} \times \frac{1}{-\cdot 0035}\left[\log(510 - \cdot 0035h) - \log 510\right]$$

whence $$\log p - \log p_0 = \frac{286}{C}\left[\log(510 - \cdot 0035h) - 6\cdot 234\right]$$

or $$\underline{\log p = \log p_0 + \frac{286}{C}\left[\log(510 - \cdot 0035h) - 6\cdot 234\right]}$$

which may be further simplified by substituting the values for p_0 and C.

General Linear Equations of the First Order, *i.e.*,

equations of the type

$$\frac{dy}{dx} + ay = b$$

where a and b may be either constants or functions of x.*

The solution of this equation may be written as

$$y = e^{-\int a dx}\{\int b e^{\int a dx} dx + C\}.$$

The proof of this rule depends upon the rule used for differentiating a product, viz., $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$; the reasoning being as follows:—

Let us first consider the simplest case in which this type of equation occurs, viz., the case of the solution of the equation

$$\frac{dy}{dx} + ay = 0$$

where a is a constant.

* See Ex. 4, p. 421, for a method of solution when b is a function of y.

Multiplying through by e^{ax} the equation becomes

$$e^{ax}\frac{dy}{dx}+aye^{ax}=0,$$

which can be written as $v\frac{dy}{dx}+y\frac{dv}{dx}=0$ (where $v=e^{ax}$ and thus $\frac{dv}{dx}=ae^{ax}$).

But $v\frac{dy}{dx}+y\frac{dv}{dx}=\frac{d}{dx}(yv)$, so that $\frac{d}{dx}(yv)=0$; hence yv must be a constant, since the result of its differentiation is to be zero.

Accordingly $\qquad yv=C,$

or $\qquad y=Cv^{-1}=Ce^{-ax}.$

Extending to the case in which b is not zero, whilst a remains a constant, *i.e.*, the case in which the equation is

$$\frac{dy}{dx}+ay=b,$$

we find that after multiplying through by e^{ax} the result arrived at is

$$\frac{d}{dx}ye^{ax}=be^{ax}.$$

Integrating both sides with regard to x,

$$ye^{ax}=\int be^{ax}dx+C$$

or
$$y=e^{-ax}\{\int be^{ax}\,dx+C\}.$$

This may be evaluated if the product of b and e^{ax} can be integrated.

For the general case, that in which a and b are functions of x, the multiplier or *integrating factor* is $e^{\int a dx}$, for after multiplication by this the equation reads

$$e^{\int a dx}\frac{dy}{dx}+ae^{\int a dx}y=be^{\int a dx}$$

and this may be written

$$\frac{d}{dx}(ye^{\int a dx})=be^{\int a dx},$$

whence by integration we find that

$$ye^{\int a dx}=\int be^{\int a dx}dx+C,$$

or
$$y=e^{-\int a dx}\{\int be^{\int a dx}\,dx+C\}.$$

Example 8.—Solve the equation $7\frac{dy}{dx}+12y=e^{5x}$.

The equation may be written

$$\frac{dy}{dx}+\frac{12}{7}y=\frac{1}{7}e^{5x}$$

so that in comparison with the standard form

$$a=\frac{12}{7} \text{ and } b=\frac{1}{7}e^{5x}.$$

Hence
$$\begin{aligned} y &= e^{-\int\frac{12}{7}dx}\left\{\int\frac{1}{7}e^{5x}.\,e^{\int\frac{12}{7}dx}\,dx+\mathrm{C}\right\} \\ &= e^{-\frac{12}{7}x}\left\{\int\frac{1}{7}e^{5x}.\,e^{\frac{12}{7}x}\,dx+\mathrm{C}\right\} \\ &= e^{-\frac{12}{7}x}\left\{\frac{1}{7}\times\frac{7}{47}e^{\frac{47}{7}x}+\mathrm{C}\right\} \\ &= \underline{\frac{1}{47}e^{5x}+\mathrm{C}e^{-\frac{12}{7}x}.} \end{aligned}$$

Example 9.—If $\frac{dy}{dx}-y=2x+1$, find an expression for y.

$$\frac{dy}{dx}-y=(2x+1) \text{ so that } a=-1,\ b=2x+1.$$

Hence
$$\begin{aligned} y &= e^{+\int dx}\{\int(2x+1)e^{-\int dx}dx+\mathrm{C}\} \\ &= e^{x}\{\int(2x+1)e^{-x}dx+\mathrm{C}\}^{*} \\ &= e^{x}\{-2xe^{-x}-3e^{-x}+\mathrm{C}\} \\ &= \underline{-2x-3+\mathrm{C}e^{x}.} \end{aligned}$$

---

* The value of the integral $\int(2x+1)e^{-x}\,dx$ is found by integrating by parts.

Thus, let $(2x+1)e^{-x}=udv$ where $dv=e^{-x}$, *i. e.*, $v=-e^{-x}$ and $u=2x+1$, *i. e.*, $\frac{du}{dx}=2$,

then
$$\begin{aligned} \int udv &= uv-\int vdu \\ &= [(2x+1)\times(-e^{-x})]-\int-e^{-x}2dx \\ &= -e^{-x}(2x+1)-2e^{-x} \\ &= -2xe^{-x}-3e^{-x}. \end{aligned}$$

Example 10.—Solve for T the equation $\frac{d\mathrm{T}}{dx}+\mathrm{PT}=\mathrm{P}(t-cx)$ (referring to the transmission of heat through cylindrical tubes); P, t and c being constants.

The equation $\frac{dT}{dx}+PT = P(t-cx)$ is of the type $\frac{dy}{dx}+ay = b$, where $a = P$ and $b = P(t-cx)$.

Hence the solution may be written

$$T = e^{-\int P dx}\{\int P(t-cx)e^{\int P dx}\,dx + K\},$$

the integrating factor being $e^{\int P dx}$, *i. e.*, e^{Px}.

Hence $$T = e^{-Px}\{\int P(t-cx)e^{Px}\,dx+K\};$$

and to express this in a simple form the integral $\int P(t-cx)e^{Px}\,dx$ must first be evaluated.

Let $\int P(t-cx)e^{Px}\,dx = \int u\,dv$ where $u = P(t-cx)$ and $dv = e^{Px}\,dx$ so that $v = \frac{1}{P}e^{Px}$; and also $du = -Pc\,dx$,

then $$\int P(t-cx)e^{Px}dx = \int u\,dv = uv - \int v\,du$$

$$= \left(P(t-cx)\times\frac{1}{P}e^{Px}\right)+\int\frac{1}{P}\times e^{Px}.Pc\,dx$$

$$= (t-cx)e^{Px}+\left(\frac{1}{P}\times ce^{Px}\right)+M.$$

$$\therefore \quad T = e^{-Px}\left\{(t-cx)e^{Px}+\frac{ce^{Px}}{P}+M+K\right\}$$

$$= \underline{(t-cx)+\frac{c}{P}+Le^{-Px}}$$

since $e^{-Px}\times e^{Px} = e^0 = 1$ and $L = M+K$.

Example 11.—When finding the currents x and y in the two coils of a wattmeter we arrive at the following differential equation;

$$\frac{dy}{dt}+\frac{R_1+R_2}{L_1+L_2}y = \frac{L_1pI}{L_1+L_2}\cos pt+\frac{R_1I}{L_1+L_2}\sin pt$$

where R_1 and L_1 are respectively the resistance and inductance of the one coil and R_2 and L_2 are the resistance and inductance of the other coil; I being the amplitude of the main current.

Solve this equation for y.

Comparing with the standard form of equation, viz., $\frac{dy}{dx}+ay=b$, we see that $a = \frac{R_1+R_2}{L_1+L_2}$ and $b = \frac{L_1pI}{L_1+L_2}\cos pt+\frac{R_1I}{L_1+L_2}\sin pt$.

Hence

$$y = e^{-\int\frac{R_1+R_2}{L_1+L_2}dt}\left\{\int\left(\frac{L_1pI}{L_1+L_2}\cos pt+\frac{R_1I}{L_1+L_2}\sin pt\right)e^{\int\frac{R_1+R_2}{L_1+L_2}dt}\,dt+C\right\}$$

$$= e^{-\frac{(R_1+R_2)t}{L_1+L_2}}\left\{\int\frac{L_1pI}{L_1+L_2}\cos pt\,.\,e^{\frac{(R_1+R_2)t}{L_1+L_2}}\,dt\right.$$

$$\left.+\int\frac{R_1I}{L_1+L_2}\sin pt\,.\,e^{\frac{(R_1+R_2)t}{L_1+L_2}}\,dt+C\right\}$$

$$= e^{-At}\left\{\int B\cos pt\,.\,e^{At}dt+\int D\sin pt\,.\,e^{At}dt+C\right\}.$$

Now, as proved on p. 157,

$$\int B\epsilon^{At}\cos pt\,dt = \frac{B\epsilon^{At}}{A^2+p^2}(p\sin pt + A\cos pt)+C_1$$

and
$$\int D\epsilon^{At}\sin pt\,dt = \frac{D\epsilon^{At}}{A^2+p^2}(A\sin pt - p\cos pt)+C_2$$

Hence

$$y = \frac{\epsilon^{-At}\times\epsilon^{At}}{A^2+p^2}\left\{pB\sin pt + AB\cos pt + DA\sin pt - Dp\cos pt + K\right\}$$

or $$y = \frac{1}{A^2+p^2}\left\{Bp\sin pt + AB\cos pt + DA\sin pt - Dp\cos pt + K\right\}$$

where $A = \dfrac{R_1+R_2}{L_1+L_2}$, $B = \dfrac{L_1pI}{L_1+L_2}$, and $D = \dfrac{R_1I}{L_1+L_2}$.

Exact Differential Equations.—An exact differential equation is one that is formed by equating an *exact differential* to zero; thus $Pdx+Qdy=0$ is the type, $Pdx+Qdy$ being an *exact differential.*

The term *exact differential* must first be explained.

$Pdx+Qdy$ is said to be an exact differential if $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$, the derivatives being partial.

To solve such an equation proceed as follows. **If the equation is exact, integrate Pdx as though y were constant, integrate the terms in Qdy that do not contain x, and put the sum of the results equal to a constant.**

[For, let $\quad Pdx+Qdy = du.$

Now, $\quad du$ is the total differential, $\dfrac{\partial u}{\partial x}dx$ and $\dfrac{\partial u}{\partial y}dy$ being the partial differentials (see p. 82);

i. e., $$du = \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy.$$

Then if $$du = 0,\ \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy = 0,$$

and this is exactly the same as the original equation

if $$\frac{\partial u}{\partial x} = P \text{ and } \frac{\partial u}{\partial y} = Q,$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y\,\partial x}$$

$$= \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial Q}{\partial x}$$

Our equation thus reduces to $du=0$, or, by integrating, to $u=C$, but $u=\int P dx$ (y being constant) $+\int Q dy$ (x being constant), and hence we have the rule as given.]

Example 12.—Solve the equation

$$(x^2-4xy-2y^2)\,dx+(y^2-4xy-2x^2)\,dy=0.$$

Here

$$P = x^2-4xy-2y^2 \qquad \frac{\partial P}{\partial y} = -4x-4y$$

$$Q = y^2-4xy-2x^2 \qquad \frac{\partial Q}{\partial x} = -4x-4y$$

and thus the equation is exact.

$\int P\,dx$ (as though y were a constant)

$$=\int(x^2-4xy-2y^2)\,dx=\frac{x^3}{3}-\frac{4x^2y}{2}-2y^2x.$$

$\int Q\,dy$ (as though x were a constant)

$$=\int(y^2-4xy-2x^2)\,dy=\frac{y^3}{3}-\frac{4xy^2}{2}-2x^2y;$$

but of this only $\frac{1}{3}y^3$ must be taken, since the other terms have been obtained by the integration of terms containing x.

Hence $\qquad \frac{1}{3}x^3-2x^2y-2xy^2+\frac{1}{3}y^3=C$

or $\qquad \underline{x^3-6x^2y-6xy^2+y^3=K.}$

Example 13.—Solve the equation $v\,du-u\,dv=0$.

If this equation is multiplied through by $\frac{1}{v^2}$ we have a form on the left-hand side with which we are familiar, viz.,

$$\frac{v\,du-u\,dv}{v^2}=0,$$

for the left-hand side is $d\left(\frac{u}{v}\right)$.

Then by integrating, $\qquad \frac{u}{v}=C$

or $\qquad u=Cv.$

[This equation might have been regarded as one made exact through multiplication by the *integrating factor* $\frac{1}{v^2}$.]

Equations Homogeneous (*i.e.*, **of the same power throughout) in x and y.**

Rule.—Make the substitution $y = vx$ and separate the variables.

Example 14.—Solve the equation $(x^2+y^2)\,dx = 2xy\,dy$.

Let $$y = vx,$$

then $$\frac{dy}{dx} = v\frac{dx}{dx}+x\frac{dv}{dx} = v+x\frac{dv}{dx} \quad \ldots\ldots \quad (1)$$

Now $$(x^2+y^2)\,dx = 2xy\,dy$$

so that $$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} = \frac{x^2+x^2v^2}{2x^2v} = \frac{1+v^2}{2v}.$$

Substituting for $\frac{dy}{dx}$ from (1)

$$v+x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

or $$x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}.$$

Separating the variables, and integrating,

$$\int\frac{2v\,dv}{1-v^2} = \int\frac{dx}{x}$$

$i.\,e.,$ $-\log(1-v^2) = \log x+\log C$ $\left\{\begin{array}{l}\text{the substitution being} \\ u = 1-v^2 \\ du = -2v\,dv\end{array}\right.$

or $\log x(1-v^2) = -\log C = \log K$

$i.\,e.,$ $x(1-v^2) = K$

or $x\left(1-\frac{y^2}{x^2}\right) = K$

and $\underline{x^2-y^2 = Kx.}$

Linear Equations of the Second Order.

Type :— $$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by = 0.$$

Let $y = e^{\lambda x}$, then $\frac{dy}{dx} = \lambda e^{\lambda x}$ and $\frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$

so that $$\lambda^2e^{\lambda x}+a\lambda e^{\lambda x}+be^{\lambda x} = 0$$

or $\lambda^2+a\lambda+b = 0$ for $y = 0$ would be a special case.

There are three possible solutions to this quadratic.

The general solution is: $$\lambda = \frac{-a \pm \sqrt{a^2-4b}}{2}$$

and let $$\lambda_1 = \frac{-a+\sqrt{a^2-4b}}{2}$$

$$\lambda_2 = \frac{-a-\sqrt{a^2-4b}}{2}$$

We shall now discuss the three cases.

Case (1).—If $a^2>4b$, then λ_1 and λ_2 are real quantities and unequal.

Now if $y=A_1e^{\lambda_1 x}$, $\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by$ will equal 0, as would be the case also if $y=A_2e^{\lambda_2 x}$, so that to complete the solution the two must be included (for the equation is true if either or both are included).

Thus $$y = A_1e^{\lambda_1 x}+A_2e^{\lambda_2 x}$$
the constants A_1 and A_2 being fixed by the conditions.

Case (2).—If $a^2 = 4b$, then $\lambda_1 = \lambda_2$.

According to the preceding case we might suppose that the solution was $$y = Ae^{\lambda x}.$$
This, however, is not the complete solution, which is $$y = (A+Bx)e^{\lambda x}.$$

Case (3).—If $a^2<4b$. This means that $\sqrt{a^2-4b}$ is the square root of a negative quantity, *i. e.*, it is an *imaginary*.

Now, $a^2-4b = -1(4b-a^2)$, $(4b-a^2)$ being positive;

hence $$\sqrt{a^2-4b} = \sqrt{-1}\sqrt{4b-a^2}$$
$$= j\sqrt{4b-a^2}$$

and $$\lambda_1 = \frac{-a+j\sqrt{4b-a^2}}{2}$$
$$\lambda_2 = \frac{-a-j\sqrt{4b-a^2}}{2}.$$

Use might be made of the solution to *Case* (1), adopting these values of λ_1 and λ_2, but this does not give the most convenient form in which to write the solution.

Let $c = \sqrt{4b-a^2}$ then $\lambda_1 = \frac{-a+jc}{2}$

and $$\lambda_2 = \frac{-a-jc}{2}$$

Then $\quad y=A_1e^{\lambda_1 x}+A_2e^{\lambda_2 x}$ from *Case* (1)

$$=A_1e^{\frac{(-a+jc)x}{2}}+A_2e^{\frac{(-a-jc)x}{2}}$$

Developing one of these only, viz., the first, and neglecting A_1 for the time being,

$$e^{\frac{-ax+jcx}{2}}=e^{-\frac{ax}{2}}\times e^{\frac{jcx}{2}}.$$

Now $\quad e^{jx}=\cos x+j\sin x$ (see p. 110),

and by writing $\frac{cx}{2}$ for x

$$e^{\frac{jcx}{2}}=\cos\frac{cx}{2}+j\sin\frac{cx}{2}$$

Hence

$$y=A_1e^{-\frac{ax}{2}}\left(\cos\frac{cx}{2}+j\sin\frac{cx}{2}\right)+A_2e^{-\frac{ax}{2}}\left(\cos\frac{cx}{2}-j\sin\frac{cx}{2}\right)$$

$$=e^{-\frac{ax}{2}}\left\{(A_1+A_2)\cos\frac{cx}{2}+j(A_1-A_2)\sin\frac{cx}{2}\right\}$$

$$=e^{-\frac{ax}{2}}A\sin\left(\frac{cx}{2}+p\right)$$

$$=Ae^{-\frac{ax}{2}}\sin\left(\frac{cx}{2}+p\right)$$

where

$$A=\sqrt{(A_1+A_2)^2+j^2(A_1-A_2)^2}=2\sqrt{A_1A_2} \quad \text{(see Part I, p. 277)}$$

and

$$\tan p=\frac{A_1+A_2}{j(A_1-A_2)}.$$

[If A_1 and A_2 are conjugate complex quantities, $\tan p$ is real.]

Thus, taking as the standard equation

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0$$

and grouping our results, we have the following:—

(1) If $a^2>4b$: the solution is

$$y=A_1e^{\frac{-a+\sqrt{a^2-4b}}{2}x}+A_2e^{\frac{-a-\sqrt{a^2-4b}}{2}x}.$$

(2) If $a^2=4b$: the solution is

$$y=(A+Bx)e^{-\frac{ax}{2}}.$$

(3) If $a^2<4b$: the solution is

$$y=Ae^{-\frac{ax}{2}}\sin\left(\frac{\sqrt{4b-a^2}}{2}x+p\right).$$

The last of these forms occurs so frequently that very careful consideration should be given to it, and to the equation of which it gives the solution.

Example 15.—Solve the equation

$$5\frac{d^2y}{dx^2}+12\frac{dy}{dx}-2y=0.$$

This can be written (after dividing through by 5)

$$\frac{d^2y}{dx^2}+2\cdot4\frac{dy}{dx}-\cdot4y=0,$$

so that $a=2\cdot4$ and $b=-\cdot4$ (in comparison with the standard form).

$$\therefore \qquad y=A_1e^{\frac{-2\cdot4+\sqrt{5\cdot76+1\cdot6}}{2}x}+A_2e^{\frac{-2\cdot4-\sqrt{5\cdot76+1\cdot6}}{2}x}$$

$$=\underline{A_1e^{\cdot16x}+A_2e^{-2\cdot56x}}.$$

It is really easier to work a question of this kind from first principles rather than to try to remember the rule in the form given; thus the values of λ will be the roots of the equation

$$5\lambda^2+12\lambda-2=0.$$

Then, calling these roots λ_1 and λ_2 respectively

$$y=A_1e^{\lambda_1x}+A_2e^{\lambda_2x}.$$

If the values of A_1 and A_2 were required, two values of y with the corresponding values of x would be necessary.

Example 16.—A body is moving away from a fixed point in such a way that its acceleration is directed towards that point, and is given in magnitude by 64 times the distance of the body from that point. Find the equation of the motion and state of what kind the motion is.

The motion is Simple Harmonic. (See p. 60.)

If $s=$ displacement at time t from the start

$$\frac{d^2s}{dt^2}=\text{acceleration and}=-64s$$

(the reason for the minus sign being obvious).

Thus $\frac{d^2s}{dt^2}+64s=0$, which is of the type $\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0$, where $a=0$ and $b=64$.

If $s=e^{\lambda t}$

$$\lambda^2+64=0$$

so that $$\lambda=\pm\sqrt{-64}=\pm8j.$$

Hence the solution (according to *Case* (3)) is

$$y = Ae^{-\frac{ax}{2}}\sin\left(\frac{cx}{2}+p\right)$$

where $a = 0$ and $c = \sqrt{4b-a^2} = 16$

so that $\underline{s = A\sin(8t+p)}$.

The general equation of S.H.M. is

$$s = A\sin(\omega t+p)$$

ω being the angular velocity, so that the angular velocity in this case is 8 and the amplitude is A.

Example 17.—Solve the equation

$$\frac{d^2y}{dx^2}+8\frac{dy}{dx}+16y = 0.$$

Let $y = e^{\lambda x}$

then $\lambda^2+8\lambda+16 = 0$

i. e., $(\lambda+4)^2 = 0$

i. e., the roots are equal.

Hence $y = (A+Bx)e^{\lambda x}$, (Cf. *Case* (2), p. 282)

where $\lambda = -4$.

hence $\underline{y = (A+Bx)e^{-4x}}$.

Example 18.—Solve the equation

$$\frac{d^2y}{dx^2}+7\frac{dy}{dx}+10y = 5.$$

This differs from the preceding examples in the substitution of a constant in place of 0.

The equation can be written

$$\frac{d^2y}{dx^2}+7\frac{dy}{dx}+10(y-\cdot 5) = 0.$$

Let $(y-\cdot 5) = e^{\lambda x}$

then $\frac{dy}{dx} = \lambda e^{\lambda x}$ and $\frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$

and $\lambda^2+7\lambda+10 = 0$

whence $\lambda = -5$ or -2

then $y-\cdot 5 = A_1e^{-5x}+A_2e^{-2x}$

or $y = A_1e^{-5x}+A_2e^{-2x}+\cdot 5$.

In other words, the solution is that of $\frac{d^2y}{dx^2}+7\frac{dy}{dx}+10y = 0$ plus $\frac{5}{10}$, *i. e.*, $\frac{\text{the constant}}{\text{coefficient of } y}$. This is correct because, if $y = \cdot 5$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ each equals 0, and thus one solution is $y = \cdot 5$. The complete solution is the sum of the two solutions.

The Operator D.—The differential coefficient of y with respect to x may be expressed in a variety of forms: thus either $\frac{dy}{dx}$, $\frac{df(x)}{dx}$, $f'(x)$ or Dy might be used to denote the process of differentiation. The last of these forms, which must only be used when there is no ambiguity about the independent variable, proves to be of great advantage when concerned with the solution of certain types of differential equations. It is found that the symbol D has many important algebraic properties, which lend themselves to the employment of D as an "operator."

The first derivative of y with regard to $x = \mathrm{D}y$, and the second derivative of y with regard to $x = \frac{d^2y}{dx^2}$, which is written as D^2y; D^2 indicating that the operation represented by D must be performed twice. This is in accordance with the ordinary rules of indices, so the fact suggests itself that the operator D may be dealt with according to algebraic rules. Thus D^3 must equal D.D.D (this implying not multiplication, but the performance of the operation three times); for

$$\mathrm{D}^3y = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\cdot\frac{d}{dx}\cdot\frac{dy}{dx} = \mathrm{D.D.D}y.$$

Our rule then holds, at any rate, so long as the index is positive, or the operation is direct; and for complete establishment we must test for the case when the index is negative.

If $\qquad \mathrm{D}y = \frac{dy}{dx} \quad \mathrm{D} = \frac{d}{dx}$: let $\frac{dy}{dx} = m$, $\quad$ *i. e.*, $\mathrm{D}y = m$.

Then by integration $\quad y = \int m\,dx$,

but if $\mathrm{D}y = m$ and the rules of algebra can be applied to D

$$y \text{ must } = \frac{m}{\mathrm{D}} \text{ or } \frac{1}{\mathrm{D}}\cdot m.$$

Hence $\qquad \frac{1}{\mathrm{D}}m = \int m\,dx$

or $\frac{1}{\mathrm{D}}$ indicates the operation of integration.

Again, if the rules of indices are to hold,

$$\mathrm{D.D}^{-1}y \text{ must } = \mathrm{D}^0y \text{ or } y,$$

hence D^{-1} must represent the process of integration; since if we

differentiate a function we must integrate the result to arrive at the original function once again.

Hence $$D^{-1} = \frac{1}{D}.$$

Having satisfied ourselves that the ordinary rules of indices may be applied to D, we may now prove that the rules of factorisation apply also.

Taking the expression $D^2-12D+32$, we can easily show that it can be written in the factor form $(D-4)(D-8)$:

for let $y = 7x^2-5x$, then $Dy = 14x-5$ and $D^2y = 14$.

Also
$$\begin{aligned}(D^2-12D+32)y &= D^2y-12Dy+32y\\ &= 14-168x+60+224x^2-160x\\ &= 224x^2-328x+74\end{aligned}$$

and
$$\begin{aligned}(D-4)(D-8)y &= (D-4)(Dy-8y)\\ &= (D-4)(14x-5-56x^2+40x)\\ &= D(14x-5-56x^2+40x)-4(14x-5-56x^2+40x)\\ &= 14-112x+40-56x+20+224x^2-160x\\ &= 224x^2-328x+74\end{aligned}$$

so that
$$(D^2-12D+32) = (D-4)(D-8).$$

It has already been shown that $\frac{1}{D}f(x) = \int f(x)dx$ and it follows that $\frac{1}{D^2}f(x) = \iint f(x)\ dx\ dx$, but the interpretation of $\frac{1}{D^2+D}f(x)$ is not seen so readily.

Now $(D^2+D)\cdot\frac{1}{(D^2+D)} = 1$ whence $(D^2+D)\cdot\frac{1}{(D^2+D)}f(x) = f(x)$

so that the operation denoted by $\frac{1}{D^2+D}$ gives a result that is annulled by the operation (D^2+D).

Example 19.—Find $\frac{108}{D^2-5D+22}e^{7x}$.

For the function e^{7x}, $D = 7$ and $D^2 = 49$, hence
$$\frac{108}{D^2-5D+22}e^{7x} = \frac{108}{49-35+22}e^{7x} = \underline{\underline{3e^{7x}}}$$

Example 20.—Find $\frac{1}{D+5}\cos 9x$
$$\frac{1}{D+5}\cos 9x = \frac{D-5}{D^2-25}\cos 9x = \frac{-9\sin 9x-5\cos 9x}{-81-25}$$
$$= \underline{\underline{\frac{9\sin 9x+5\cos 9x}{106}}}.$$

Example 21.—Find $\frac{2}{D+2}(x^3-5x^2+11x-8)$

$$\frac{2}{D+2} = \frac{2}{2\left(1+\frac{D}{2}\right)} = \left(1+\frac{D}{2}\right)^{-1} = 1-\frac{D}{2}+\frac{D^2}{4}-\frac{D^3}{8}\ldots$$

so that

$$\begin{aligned}\frac{2}{D+2}(x^3-5x^2+11x-8) &= \left(1-\frac{D}{2}+\frac{D^2}{4}-\frac{D^3}{8}\ldots\right)(x^3-5x^2+11x-8)\\ &= (x^3-5x^2+11x-8)-\left(\frac{3x^2}{2}-5x+\frac{11}{2}\right)\\ &\quad+\left(\frac{3x}{2}-\frac{5}{2}\right)-\frac{3}{4}\\ &= \underline{x^3-\frac{13x^2}{2}+\frac{35x}{2}-\frac{67}{4}}.\end{aligned}$$

The "Shift" Theorems.—Products in which e^{ax} is a factor may be differentiated and integrated by the use of the following devices:—

$$f(D)\,e^{ax}\,F(x) = e^{ax}f(D+a)\,F(x)$$

and
$$\frac{1}{f(D)}\,e^{ax}\,F(x) = e^{ax}\frac{1}{f(D+a)}\,F(x).$$

Example 22.—Find $(D^2-2D+5)e^{3x}x^5$.

$$\begin{aligned}(D^2-2D+5)e^{3x}x^5 &= e^{3x}\{(D+3)^2-2(D+3)+5\}x^5 = e^{3x}(D^2+4D+8)x^5\\ &= e^{3x}(20x^3+20x^4+8x^5) = \underline{4x^3e^{3x}(2x^2+5x+5)}.\end{aligned}$$

Example 23.—Find $\int e^{-2x}\sin 7x\,dx$

$$\begin{aligned}\int e^{-2x}\sin 7x\,dx = \frac{1}{D}e^{-2x}\sin 7x &= e^{-2x}\frac{1}{D-2}\sin 7x\\ &= e^{-2x}\frac{(D+2)}{D^2-4}\sin 7x\\ &= \frac{e^{-2x}(7\cos 7x+2\sin 7x)}{-49-4}\\ &= \underline{-\frac{e^{-2x}(7\cos 7x+2\sin 7x)}{53}}.\end{aligned}$$

Example 24.—Find $\frac{1}{D-3}e^{3x}$

If 3 is substituted for D the denominator becomes zero and what is known as a "case of failure" arises. The difficulty is surmounted by applying the shift theorem, thus :—

$$\frac{1}{D-3}e^{3x}1 = e^{3x}\frac{1}{(D+3)-3}1 = e^{3x}\frac{1}{D}1 = \underline{xe^{3x}}.$$

Application of these Rules to the Solution of Differential Equations.

Example 25.—Solve the equation $\frac{d^2y}{dx^2}+7\frac{dy}{dx}+12y = e^{5x}$.

This equation might be written

$$(D^2+7D+12)y = e^{5x}$$

so that $$y = \frac{e^{5x}}{D^2+7D+12}.$$

The solution of this equation gives the particular integral, whilst the *complementary function*, as it is termed, will be obtained by the solution of the equation

$$\frac{d^2y}{dx^2}+7\frac{dy}{dx}+12y = 0.$$

$$(D+3)(D+4)y = 0 \text{ so that } D = -3 \text{ or } -4$$
$$\text{and } y = A_1e^{-3x}+A_2e^{-4x}$$

Now $De^{5x} = 5e^{5x}$, $D^2e^{5x} = 25e^{5x}$, *i.e.*, $D = 5$ and $D^2 = 25$.

Hence the particular integral is

$$y = \frac{e^{5x}}{25+35+12}$$
$$= \frac{e^{5x}}{72}$$

and the general solution is

$$\underline{y = A_1e^{-3x}+A_2e^{-4x}+\frac{e^{5x}}{72}}$$

To test this by differentiation of the result:—

$$y = A_1e^{-3x}+A_2e^{-4x}+\frac{e^{5x}}{72}$$
$$\frac{dy}{dx} = -3A_1e^{-3x}-4A_2e^{-4x}+\frac{5}{72}e^{5x}$$
$$\frac{d^2y}{dx^2} = 9A_1e^{-3x}+16A_2e^{-4x}+\frac{25}{72}e^{5x}$$

$$\therefore \quad \frac{d^2y}{dx^2}+7\frac{dy}{dx}+12y = 9A_1e^{-3x}+16A_2e^{-4x}+\frac{25}{72}e^{5x}$$
$$-21A_1e^{-3x}-28A_2e^{-4x}+\frac{35}{72}e^{5x}$$
$$+12A_1e^{-3x}+12A_2e^{-4x}+\frac{12}{72}e^{5x}$$
$$= e^{5x}.$$

Example 26.—Solve the equation $\frac{d^2s}{dt^2}+4\frac{ds}{dt}+4s = 5 \sin 7t$. (This type of equation occurs frequently in electrical problems and in problems on forced vibrations of a system.)

The solution of $$\frac{d^2s}{dt^2}+4\frac{ds}{dt}+4s=0$$

is $$s=(A+Bt)e^{-2t}$$ (See p. 283.)

To find the particular integral:—

$$(D^2+4D+4)s=5\sin 7t.$$

$$\therefore \qquad s=\frac{5\sin 7t}{D^2+4D+4}.$$

$$D\sin 7t=7\cos 7t \quad\text{and}\quad D^2\sin 7t=-49\sin 7t.$$

(Note that $D^2=-49$, but D does not $=7$.)

We must thus eliminate D from the denominator: to do this, we first replace D^2 by -49

$$\begin{aligned}\text{Then}\qquad s&=\frac{5\sin 7t}{-49+4D+4}\\&=\frac{5(4D+45)\sin 7t}{16D^2-2025}\\&=\frac{-5(45\sin 7t+28\cos 7t)}{2809}\\&=\frac{-5\times 53\sin\left(7t+\tan^{-1}\frac{28}{45}\right)}{2809}\\&=-\frac{5}{53}\sin\left(7t+\tan^{-1}\frac{28}{45}\right).\end{aligned}$$

Hence the complete solution is

$$\underline{s=(A+Bt)e^{-2t}-\frac{5}{53}\sin\left(7t+\tan^{-1}\frac{28}{45}\right).}$$

Example 27.—Find $\frac{1}{D^2+a^2}\sin ax$.

Substitution of $-a^2$ for D^2 leads to a case of failure and an entirely different approach must be made.

For D^2+a^2 write $D^2-j^2a^2$ *i.e.* $(D-ja)(D+ja)$ and express $\sin ax$ in its exponential form, $\frac{e^{jax}-e^{-jax}}{2j}$.

Then

$$\begin{aligned}\frac{1}{D^2+a^2}\sin ax&=\frac{1}{2j}\left\{\frac{e^{jax}}{(D-ja)(D+ja)}-\frac{e^{-jax}}{(D-ja)(D+ja)}\right\}\\&=\frac{1}{2j}\left\{\frac{e^{jax}}{(D+ja)(ja+ja)}-\frac{e^{-jax}}{(-ja-ja)(D+ja)}\right\}\\&=-\frac{1}{4a}\left(\frac{e^{jax}}{D-ja}+\frac{e^{-jax}}{D+ja}\right)\end{aligned}$$

$$= -\frac{1}{4a}\left(e^{jax}\frac{1}{D}1 + e^{-jax}\frac{1}{D}1\right)$$

$$= -\frac{x}{2a}\left(\frac{e^{jax}+e^{-jax}}{2}\right)$$

$$= -\frac{x\cos ax}{2a}.$$

In like manner it could be shown that

$$\frac{1}{D^2+a^2}\cos ax = \frac{x\sin ax}{2a}.$$

Thus $\frac{1}{D^2+9}(12\cos 3x - 15\sin 3x) = x(2\sin 3x + \frac{5}{2}\cos 3x)$.

Example 28.—The equation $EI\frac{d^2y}{dx^2} + Fy + \frac{Bl}{8}\cos\frac{\pi x}{l} = 0$ occurs in Mechanics, y being the deflection of a rod of length l, and F being the end load.

Solve this equation.

We may rewrite the equation as $\frac{d^2y}{dx^2} + \frac{F}{EI}y = -\frac{Bl}{8EI}\cos\frac{\pi x}{l}$ . . (1)

The solution of $\frac{d^2y}{dx^2} + \frac{F}{EI}y = 0$ is $y = A\sin\left(\sqrt{\frac{F}{IE}}x + p\right)$ . . . (2)

Reverting to form (1)

$$D^2y + \frac{F}{EI}y = -\frac{Bl}{8EI}\cos\frac{\pi x}{l}$$

or
$$y = \frac{-\frac{Bl}{8EI}\cos\frac{\pi x}{l}}{D^2+\frac{F}{EI}}$$

$$= -\frac{Bl}{8IE}\cos\frac{\pi x}{l} \times \frac{1}{-\frac{\pi^2}{l^2}+\frac{F}{EI}} \quad \text{for } D^2 = -\frac{\pi^2}{l^2}$$

$$= \frac{\frac{Bl}{8}\cos\frac{\pi x}{l}}{\frac{EI\pi^2}{l^2} - F}.$$

Hence the complete solution is $y = A\sin\left(\sqrt{\frac{F}{IE}}x + p\right) + \frac{\frac{Bl}{8}\cos\frac{\pi x}{l}}{\frac{EI\pi^2}{l^2} - F}$.

Example 29.—A pin-jointed column, initially bent to a curve of cosines, has a vertical load W applied to it. Find an expression for the deflection at any point.

Given that the equation of the initial bent form is

$$y_1 = \mathrm{A}\cos\left(\frac{\pi x}{l}\right)$$

y being the deflection at distance x from the centre of the column, which is of length l.

Also $$-\mathrm{EI}\frac{d^2y}{dx^2} = \mathrm{W}\left\{y+\mathrm{A}\cos\left(\frac{\pi x}{l}\right)\right\}$$

this equation being obtained from a consideration of the bending moment at distance x from the centre.

$$-\mathrm{EI}\frac{d^2y}{dx^2} = \mathrm{W}\left\{y+\mathrm{A}\cos\left(\frac{\pi x}{l}\right)\right\}$$

and thus $$\frac{d^2y}{dx^2}+\frac{\mathrm{W}}{\mathrm{EI}}\left\{y+\mathrm{A}\cos\left(\frac{\pi x}{l}\right)\right\} = 0$$

or $$\frac{d^2y}{dx^2}+\frac{\mathrm{W}}{\mathrm{EI}}y = -\frac{\mathrm{AW}}{\mathrm{EI}}\cos\left(\frac{\pi x}{l}\right).$$

Now, as shown on p. 283, the solution of the equation

$$\frac{d^2y}{dx^2}+\frac{\mathrm{W}}{\mathrm{EI}}y = 0$$

is $$y = \mathrm{B}\sin\left(\sqrt{\frac{\mathrm{W}}{\mathrm{EI}}}x+p\right).$$

To find the particular integral, viz., the solution of the equation

$$\frac{d^2y}{dx^2}+\frac{\mathrm{W}}{\mathrm{EI}}y = -\frac{\mathrm{AW}}{\mathrm{EI}}\cos\left(\frac{\pi x}{l}\right)$$

write the equation as

$$\left(\mathrm{D}^2+\frac{\mathrm{W}}{\mathrm{EI}}\right)y = -\frac{\mathrm{AW}}{\mathrm{EI}}\cos\left(\frac{\pi x}{l}\right)$$

so that $$y = \frac{-\frac{\mathrm{AW}}{\mathrm{EI}}\cos\left(\frac{\pi x}{l}\right)}{\mathrm{D}^2+\frac{\mathrm{W}}{\mathrm{EI}}}$$

$$= \frac{-\frac{\mathrm{AW}}{\mathrm{EI}}\cos\left(\frac{\pi x}{l}\right)}{\frac{\mathrm{W}}{\mathrm{EI}}-\frac{\pi^2}{l^2}}.$$

Combining the two results

$$\underline{y = \mathrm{B}\sin\left(\sqrt{\frac{\mathrm{W}}{\mathrm{EI}}}x+p\right)+\frac{l^2\mathrm{AW}\cos\left(\frac{\pi x}{l}\right)}{\pi^2\mathrm{EI}-\mathrm{W}l^2}.}$$

The following example combines the methods of solution employed in *Examples* 17, 18, 19 and 20.

Example 30.—Solve the equation

$$\frac{d^2s}{dt^2}-12\frac{ds}{dt}+20s = e^{-5t}+\sin 6t+5.$$

(*a*) The solution of

$$\frac{d^2s}{dt^2}-12\frac{ds}{dt}+20s=0 \text{ is } s=A_1e^{10t}+A_2e^{2t}.$$

(*b*) The particular integral for

$$\frac{d^2s}{dt^2}-12\frac{ds}{dt}+20s=e^{-5t}$$

may be thus found :—

$$s=\frac{1}{D^2-12D+20}e^{-5t} \qquad De^{-5t}=-5e^{-5t}$$
$$D^2e^{-5t}=25e^{-5t}$$

$$=\frac{1}{25+60+20}\cdot e^{-5t}$$

$$=\frac{e^{-5t}}{105}$$

(*c*) To find the particular integral for

$$\frac{d^2s}{dt^2}-12\frac{ds}{dt}+20s=\sin 6t$$

$$s=\frac{1}{D^2-12D+20}\sin 6t=\frac{1}{-36-12D+20}\sin 6t$$

$$=-\frac{1}{4}\frac{1}{(3D+4)}\sin 6t \quad =-\frac{1}{4}\frac{(3D-4)}{(9D^2-16)}\sin 6t$$

$$=-\frac{1}{4}\frac{(18\cos 6t-4\sin 6t)}{-340}$$

$$=\frac{9\cos 6t-2\sin 6t}{680}.$$

(*d*) The particular integral for

$$\frac{d^2s}{dt^2}-12\frac{ds}{dt}+20s=5$$

is

$$s=\frac{5}{20}=\frac{1}{4}$$

Hence the complete solution is the sum of those in (*a*), (*b*), (*c*) and (*d*), viz.,

$$s=A_1e^{10t}+A_2e^{2t}+\frac{e^{-5t}}{105}+\frac{9\cos 6t-2\sin 6t}{680}+\frac{1}{4}.$$

Equations of the Second Degree.—The treatment of these equations is very similar, up to a certain point, to that employed in the solution of ordinary quadratic equations; particularly the solution by factorisation.

Example 31.—Solve the equation

$$\left(\frac{dy}{dx}\right)^2 - 8\frac{dy}{dx} - 33 = 0.$$

Let $\quad Y = \frac{dy}{dx} \quad$ then $\quad Y^2 - 8Y - 33 = 0$

$\qquad\qquad\qquad\qquad$ or $\quad (Y-11)(Y+3) = 0.$

Thus $\quad Y = 11 \quad$ or $\quad Y = -3$

i.e., $\quad \frac{dy}{dx} = 11 \quad$ or $\quad \frac{dy}{dx} = -3,$

whence $y = 11x + C_1 \quad$ or $\quad y = -3x + C_2$

or $\quad y - 11x - C_1 = 0 \qquad\quad y + 3x - C_2 = 0,$

and the complete solution is the *product* of these two solutions, since the equation is of "degree" higher than the first.

Thus the solution is

$$\underline{(y - 11x - C_1)(y + 3x - C_2) = 0.}$$

Example 32.—Solve the equation $5\left(\frac{dy}{dx}\right)^2 - 8y^3 = 0.$

$$5\left(\frac{dy}{dx}\right)^2 - 8y^3 = 0.$$

Dividing by 5 $\qquad \left(\frac{dy}{dx}\right)^2 - 1\cdot6y^3 = 0.$

Factorising $\qquad \left(\frac{dy}{dx} + 1\cdot265y^{\frac{3}{2}}\right)\left(\frac{dy}{dx} - 1\cdot265y^{\frac{3}{2}}\right) = 0.$

Hence $\qquad \frac{dy}{dx} + 1\cdot265y^{\frac{3}{2}} = 0 \quad$ or $\quad \frac{dy}{dx} - 1\cdot265y^{\frac{3}{2}} = 0.$

Separating the variables and integrating

$$\int\frac{dy}{y^{\frac{3}{2}}} + 1\cdot265\int dx = 0 \quad \text{or} \quad \int\frac{dy}{y^{\frac{3}{2}}} - 1\cdot265\int dx = 0$$

$$-\frac{2}{y^{\frac{1}{2}}} + 1\cdot265x = C_1 \quad \text{or} \quad -\frac{2}{y^{\frac{1}{2}}} - 1\cdot265x = C_2$$

whence the complete solution is,

$$\underline{\left(1\cdot265x - C_1 - \frac{2}{y^{\frac{1}{2}}}\right)\left(-1\cdot265x - C_2 - \frac{2}{y^{\frac{1}{2}}}\right) = 0.}$$

Two further examples are added to illustrate methods of solution other than those already indicated.

Example 33.—Solve the equation $\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-14\frac{dy}{dx}+24y = 0$.

The equation may be written $(D^3-D^2-14D+24)y = 0$

or $(D-2)(D-3)(D+4)y = 0,$

whence $\underline{y = Ae^{2x}+Be^{3x}+Ce^{-4x}.}$

Example 34.—The equation $\frac{d^4y}{dx^4} = m^4y$ occurs in the discussion of the whirling of shafts. Solve this equation.

$$\frac{d^4y}{dx^4} = m^4y,$$

i. e., $D^4 = m^4$

or $(D^2-m^2)(D^2+m^2) = 0,$

whence $D = \pm m$ or $\pm jm.$

Hence $y = a_1e^{mx}+a_2e^{-mx}+a_3e^{jmx}+a_4e^{-jmx}.$

But $e^{jx} = \cos x+j\sin x,\ e^{jmx} = \cos mx+j\sin mx$

and $e^x = \cosh x+\sinh x,$

i. e., $e^{mx} = \cosh mx+\sinh mx.$

$$\begin{aligned}\therefore\quad y &= a_1(\cosh mx+\sinh mx)+a_2(\cosh mx-\sinh mx)\\ &\quad +a_3(\cos mx+j\sin mx)+a_4(\cos mx-j\sin mx)\\ &= (a_3+a_4)\cos mx+(a_3-a_4)j\sin mx+(a_1+a_2)\cosh mx\\ &\quad +(a_1-a_2)\sinh mx\\ &= \underline{A\cos mx+B\sin mx+C\cosh mx+D\sinh mx.}\end{aligned}$$

The constants A, B, C and D are found by consideration of the conditions; four equations must be formed, these being found by successive differentiation and by substituting for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ their values for various values of x.

Exercises 22.—On the Solution of Differential Equations.

1. If $\frac{dy}{dx} = 5x^2-2{\cdot}4$ and $y = 1{\cdot}68$ when $x = 2{\cdot}29$, find y in terms of x.

2. Given that $\frac{d^2s}{dt^2} = 16{\cdot}1$; $\frac{ds}{dt} = 4{\cdot}3$ when $t = 1{\cdot}7$ and $s = 9{\cdot}8$ when $t = {\cdot}2$, find s in terms of t.

3. If $\frac{dy}{dx} = 8y+5$, find an expression for y

4. Given that $8{\cdot}76\frac{dy}{dx}+9{\cdot}15y=76{\cdot}4$ and also that $y=2{\cdot}17$ when $x=0$, find an expression for y in terms of x.

5. A beam simply supported at its ends is loaded with a concentrated load W at the centre. The bending moment M at a section distant x from the centre is given by

$$M=\frac{W}{2}\left(\frac{l}{2}-x\right).$$

If $\frac{M}{EI}=\frac{d^2y}{dx^2}$, find the equation of the deflected form, y being the deflection.

6. For the case of a fixed beam uniformly loaded, M, the bending moment, $=\left\{\frac{w}{2}\left(\frac{l^2}{4}-x^2\right)-K\right\}$. If $\frac{M}{EI}=\frac{d^2y}{dx^2}$; $\frac{dy}{dx}=0$ when $x=\frac{l}{2}$ and also when $x=0$; and $y=0$ when $x=\frac{l}{2}$, find an expression for y.

7. Solve the equation

$$\frac{\pi r D}{as(1+r)}\frac{dx}{dT}=\frac{1}{T-t},$$

the limits to T being T_1 and T_2, and to x being 0 and l; the remaining letters representing constants.

8. If $\frac{dv}{dx}=-\frac{xp}{4l\mu}$ and $v=0$ when $x=s$, find v in terms of x.

9. If $\tau_1=$ the absolute temperature of the gases entering a tube of length l and diam. D,
$\tau_2=$ the absolute temperature of the gases leaving this tube,
$\theta=$ temperature of the water,
$Q=$ amount of heat transmitted through the tube per sq. ft. per sec. per degree difference of temperature on the two sides,
$w=$ weight of gases along the tube per sec., and $s=$ specific heat of gases,

then $$\frac{d\tau}{\tau-\theta}+\frac{Q\pi D dx}{ws}=0.$$

Find an expression for Q, x being the distance from one end of the tube.

10. Find Q if $Q\tau^2\pi D dx+ws\,d\tau=0$ (the letters having the same meanings as in No. 9, and the limits being the same).

11. If τ_1 and τ_2 are the inside and outside temperatures respectively of a thick tube of internal radius r_1 and external radius r_2, then

$$\frac{d\tau}{dx}=-\frac{H}{2\pi K(r_1+x)l}.$$

l is the length of the tube, H is a quantity of heat, and the limits to x are 0 and r_2-r_1. Find an expression for H, K being a constant.

12. A compound pendulum swings through small arcs. If I = moment of inertia about the point of suspension, h = the distance of C. of G. from point of suspension, then

$$\mathrm{I} \times \text{angular acceleration} \left(i.\ e.,\ \mathrm{I}\frac{d^2\theta}{dt^2}\right) = -mh\theta.$$

Find an expression for θ.

If μ = couple for unit angle = mh, prove that t, the time of a complete oscillation, $= 2\pi\sqrt{\frac{\mathrm{I}}{\mu g}}$ (in Engineers' units).

13. To find an expression for the stress p in thick cylinders it is necessary to solve the equation

$$\frac{rdp}{dr}+2p = 2\mathrm{A}.$$

Solve this equation for p.

14. For a thick spherical shell, if p = radial pressure,

$$3a+3p = -x\frac{dp}{dx}.$$

Find an expression for p, a being a constant.

15. If $-\mathrm{K}_v vdp = \mathrm{K}_p pdv$, prove that pv^γ = constant, γ being the ratio $\frac{\text{the specific heat at constant pressure}}{\text{the specific heat at constant volume}}$ of a gas, $i.\ e.$, $\frac{\mathrm{K}_p}{\mathrm{K}_v}$.

16. Solve for z in the equation

$$\frac{dz}{dx}+\frac{w}{g}\frac{\omega^2}{f}zx = 0.$$

17. Solve the equation

$$\frac{d^2y}{dx^2}-17\frac{dy}{dx}+70y = 0$$

and thence the equation

$$\frac{d^2y}{dx^2}-17\frac{dy}{dx}+70y = 70.$$

18. Solve the equation $\qquad \frac{d^2s}{dt^2} = 87s.$

19. Solve the equation $\qquad \frac{d^2s}{dt^2}+87s = 0.$

20. Find the time that elapses whilst a capacitor of capacitance K discharges through a constant resistance R, the potential difference at the start being v_1 and at the end v_2, being given that

$$-\mathrm{K} \times \text{rate of change of potential } v = \frac{v}{\mathrm{R}}.$$

21. If $\mathrm{V} = \mathrm{RI}+\mathrm{L}\frac{dI}{dt}$, and V = 0, find an expression for I ; I_0 being the initial current, $i.\ e.$, the value of I when $t = 0$.

22. If $\mathrm{V} = \mathrm{RI}+\mathrm{L}\frac{dI}{dt}$, and $\mathrm{V} = \mathrm{V}_0 \sin \omega t$, find an expression for I.

23. If $\frac{dy}{dx}=\frac{1}{a}\sqrt{y^2+2ay}$, find x in terms of y, a being a constant.

24. An equation occurring when considering the motion of the piston of an indicator is

$$M\frac{d^2x}{dt^2}+\frac{a}{SM}x=\frac{pa}{M}.$$

Solve this equation for x; M, a, S and p being constants.

25. If
$$-Py=EI\frac{d^2y}{dx^2}$$
(an equation referring to the bending of struts), find y; given that $x=0$ when $y=0$, and $y=Y$ when $x=\frac{L}{2}$.

26. To find the time t of the recoil of a gun, it was necessary to solve the equation $\frac{dx}{dt}=n\sqrt{x^2-a^2}$.

If $a=47{\cdot}5$, $n=3{\cdot}275$ and the limits to x are 0 and $57{\cdot}5$, find t.

27. Solve the equation $\frac{d^2x}{dt^2}+2f\frac{dx}{dt}+n^2x=0$.

Take $n^2=200$, $f=7{\cdot}485$; also let $x=0$ and $\frac{dx}{dt}=10$ when $t=0$.

28. The equation $\frac{d^2x}{dt^2}+2f\frac{dx}{dt}+n^2x=a\sin qt$ expresses the forced vibration of a system. If $n^2=49$, $f=3$, $q=5$, find an expression for x.

29. Solve the equation $\frac{d^2V}{dx^2}-V\frac{r_1}{r_2}=0$.

30. If H is the amount of heat given to a gas, p is its pressure and v its volume (of 1 lb.), then $\frac{dH}{dv}=\frac{1}{(n-1)}\left(v\frac{dp}{dv}+np\right)$. Assuming that there is no change of heat $\left(i.e.,\text{ the expansion is adiabatic and }\frac{dH}{dv}=0\right)$, find a simple equation to express the connection between the pressure and volume during this expansion.

31. Newton's law of cooling may be expressed by the equation

$$\frac{d\theta}{dt}=-k(\theta-\theta_a)$$

where k is a constant, and θ_a is the temperature of the air.

If $\theta=\theta_0$ when $t=0$, find an expression for θ.

32. The equation $1{\cdot}04\frac{d^2\theta}{dt^2}+12{\cdot}3\frac{d\theta}{dt}+13\theta-634=0$ occurred in an investigation to find θ, the angle of incidence of the main planes of an aeroplane.

If $t=0$ when $\theta=1$ and $\frac{d\theta}{dt}=0$ when $t=0$, find an expression for θ.

33. A circular shaft weighing ρ lbs. per ft. rotates at ω radians per second, and is subjected to an endlong compressive force F The deflection y can be found from the equation

$$\frac{d^4y}{dx^4}+\frac{\text{F}}{\text{EI}}\frac{d^2y}{dx^2}-\frac{\rho}{g}\frac{\omega^2 y}{\text{EI}}-\frac{\rho}{\text{EI}}=0.$$

Solve this equation for y.

34. An equation relating to the theory of the stability of an aeroplane is

$$\frac{dv}{dt}=g\cos a-kv$$

where v is a velocity; g, a and k being constants. Find an expression for the velocity, if it is known that $v=0$ when $t=0$.

35. If $\dfrac{dv}{dt}=\dfrac{a}{25}(25-v^2)$ find an expression for t.

36. Solve the equation $2\dfrac{d^2y}{dx^2}-60\dfrac{dy}{dx}+400y=0$, it being given that $y=5$ when $x=0$ and $y=27{\cdot}6$ when $x={\cdot}1$.

37. Solve the equations $\dfrac{dy}{dx}-5y=e^{5x}$ and $\dfrac{dy}{dx}-5y=e^{4x}$ and note the vital difference in the form of the solution.

38. An equation important in the treatment of the vibration of membranes is

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0.$$

Show that this is satisfied by

$$y=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\frac{x^8}{2^2.4^2.6^2.8^2}-\ldots.$$

CHAPTER X

APPLICATIONS OF THE CALCULUS

THE idea of this chapter is to illustrate the use of the Calculus as applied to many Engineering problems; and the reader is supposed to be acquainted with the technical principles involved. The various cases will be dealt with as though examples.

Examples in Thermodynamics.

Example 1.—To prove that $(V-w)=\frac{L}{\tau}\frac{d\tau}{dP}$, an equation occurring in Thermodynamics,

where $L=$ latent heat at absolute temperature τ,

$V=$ vol. of 1 lb. of steam at absolute temperature τ,

$w=$ vol. of 1 lb. of water $=\cdot 016$ cu. ft.,

$P=$ pressure.

A quantity q of heat taken in at $\tau+\delta\tau$ and discharged at τ will, according to the Carnot cycle, give out work $=q\frac{\tau+\delta\tau-\tau}{\tau+\delta\tau}$ or approximately $q\frac{\delta\tau}{\tau}$.

Hence for 1 lb. of steam at the boiling temperature,

$$\text{work}=q\frac{\delta\tau}{\tau}=L\frac{\delta\tau}{\tau},$$

but the work done = volume of steam in the cylinder × change in pressure

$$=(V-w)\times\delta P.$$

Hence $$(V-w)\delta P=L\frac{\delta\tau}{\tau},$$

and thus $$V-w=\frac{L}{\tau}\frac{\delta\tau}{\delta P},$$

or, as $\delta\tau$ becomes infinitely small,

$$V-w=\frac{L}{\tau}\frac{d\tau}{dP}.$$

Now $\frac{d\tau}{dP}$ is the slope of the pressure-temperature curve (plotted from

the tables) and can be easily found for any temperature τ. Hence V can also be found.

A numerical example will illustrate further.

It is required to find the volume of 1 lb. of dry steam at 228° F., *i. e.*, at 20 lbs. per sq. in. pressure.

From the steam tables $\begin{cases} \text{When } P = 19, & t = 225{\cdot}3, & \tau = 460+225{\cdot}3 = 685{\cdot}3. \\ \quad\text{,,}\quad P = 20, & t = 228, & \tau = 688. \\ \quad\text{,,}\quad P = 21, & t = 230{\cdot}6, & \tau = 690{\cdot}6. \end{cases}$

Plotting these temperatures to a base of pressures, we find that the portion of the curve dealt with in this range is practically straight.

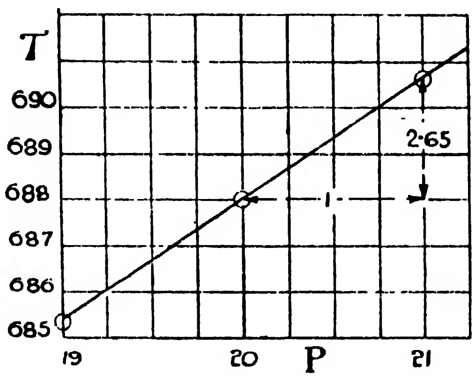


FIG. 103.—Problem in Thermodynamics.

The slope of this line = 2·65 (Fig. 103), and this is the value of $\frac{d\tau}{dP}$, P being given in lbs. per sq. in. and the latent heat in thermal units. To change the formula to agree with these units,

$$V = w + \frac{778L}{144\tau}\frac{d\tau}{dP}.$$

Also L at 228° F. = 953.

$$\therefore \qquad V = {\cdot}016 + \frac{778 \times 953 \times 2{\cdot}65}{144 \times 688}$$

$$= \underline{19{\cdot}82 \text{ cu. ft.}}$$

Example 2.—To prove that the specific heat of saturated steam (expanding dry) is negative.

Let Q = the quantity of heat added.

H = total heat from 32° F.

I = internal energy of the steam.

Then $\quad H = \text{internal} + \text{external energy}$

$\qquad = I+PV,$

i.e., $\quad I = H-PV$

or $\quad \delta I = \delta H-\delta(PV) = \delta H-(P\delta V+V\delta P).$

Now $\quad \delta Q = \delta I+P\delta V$

$\qquad = \delta H-P\delta V-V\delta P+P\delta V$

$\qquad = \delta H-V\delta P$

$\qquad = \delta H-\frac{L}{\tau}\delta\tau \quad \left\{\begin{array}{l}\text{from } \textit{Example } 1, \text{ neglecting} \\ w, \text{ which is very small}\end{array}\right\}.$

Then $\quad \frac{\delta Q}{\delta\tau} = \frac{\delta H}{\delta\tau}-\frac{L}{\tau}$

and $\quad \frac{dQ}{d\tau} = \frac{dH}{d\tau}-\frac{L}{\tau} \quad \left\{\begin{array}{l} H = 1082+\cdot305\tau \\ \therefore \frac{dH}{d\tau} = \cdot305\end{array}\right.$

$\qquad = \cdot305-\frac{L}{\tau}$

Now the specific heat = heat to raise the temperature 1°

$$= \frac{dQ}{d\tau}$$

hence $\quad s = \cdot305-\frac{L}{\tau}$

and since $\frac{L}{\tau}$ is greater than ·305, s is a negative quantity.

E.g., if $\quad t = 300° \text{ F., i. e., } \tau = 761° \text{ F. absol.,}$

$\qquad L = 1115-\cdot7t$

$\qquad = 1115-210 = 905.$

$\qquad \therefore s = \cdot305-\frac{905}{761}$

$\qquad = -\cdot882.$

Work Done in the Expansion of a Gas.

Example 3.—Find the work done in the expansion of a gas from volume v_1 to volume v_2.

There are two distinct cases, which must be treated separately; but for both cases the work done in the expansion is measured by the area

$$ABCD = \Sigma \text{ areas of strips like MN} \quad \text{(Fig. 104)}$$

$$\doteq \Sigma_{v_1}^{v_2} p\,\delta v \quad \text{or} \quad \int_{v_1}^{v_2} p\,dv \quad \text{more exactly.}$$

Case (*a*), for which the law of the expansion is $pv = C$.

Work done $\quad = \int_{v_1}^{v_2} p\,dv = \int_{v_1}^{v_2} Cv^{-1}\,dv = C\Big(\log v\Big)_{v_1}^{v_2}$

$$= C\log\left(\frac{v_2}{v_1}\right)$$

$$\text{or} \quad C\log r$$

r being the ratio of expansion, and equal to $\frac{v_2}{v_1}$.

Thus the work done $= \underline{pv\log r}$.

Case (*b*), for which the law of the expansion is $pv^n = C$, n having any value other than 1.

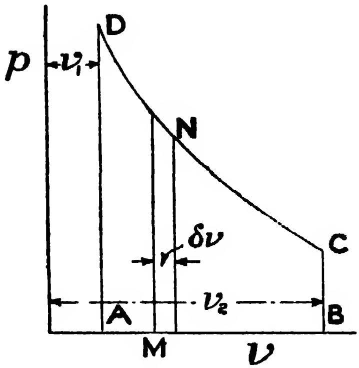


FIG. 104.

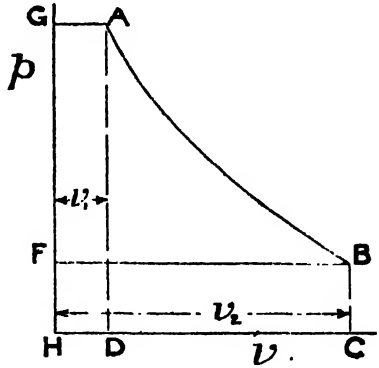


FIG. 105.

Work Done in the Expansion of a Gas.

$$\text{Work done} = \int_{v_1}^{v_2} p\,dv = \int_{v_1}^{v_2} Cv^{-n}\,dv$$
$$= \frac{C}{1-n}(v_2^{1-n} - v_1^{1-n})$$
$$= \frac{1}{1-n}(Cv_2^{1-n} - Cv_1^{1-n})$$
$$= \frac{1}{1-n}(p_2v_2^nv_2^{1-n} - p_1v_1^nv_1^{1-n})$$
$$= \frac{p_2v_2 - p_1v_1}{1-n}.$$

Work Done in a Complete Theoretical Cycle.

Example 4.—Find the work done in the complete cycle represented by the diagram FGAB in Fig. 105.

$$\text{The work done} = \text{area GABF} = \text{ABCD} + \text{GADH} - \text{FBCH}$$
$$= \frac{p_2v_2 - p_1v_1}{1-n} + p_1v_1 - p_2v_2$$
$$= (p_1v_1 - p_2v_2)\left(1 - \frac{1}{1-n}\right)$$
$$= (p_1v_1 - p_2v_2)\frac{(-n)}{1-n}$$
$$= \frac{p_1v_1 - p_2v_2}{\frac{n-1}{n}}.$$

Note that, if $n=\frac{17}{16}$

work done $=\dfrac{p_1v_1-p_2v_2}{\frac{1}{17}}$

$= 17\,(p_1v_1-p_2v_2).$

If the expansion is adiabatic, and n is calculated according to Zeuner's rule, $n = 1{\cdot}035+{\cdot}1q$

(q being the initial dryness fraction).

If $q = 1$, then $n = 1{\cdot}035+{\cdot}1 = 1{\cdot}135$, so that the work done

$$= \frac{1{\cdot}135}{{\cdot}135}(p_1v_1-p_2v_2) = 8{\cdot}41\ (p_1v_1-p_2v_2).$$

To Find the Entropy of Water at Absolute Temperature τ.

Example 5.—When a substance takes in or rejects heat (at temperature τ) the change in entropy $\delta\phi = \frac{\delta q}{\tau}$ (δq = heat taken in).

Let σ = specific heat,

then $\sigma\delta\tau = \delta q$.

$$\text{Change in entropy from } \tau_0 \text{ to } \tau = \int_{\tau_0}^{\tau}\frac{dq}{\tau}$$

$$= \sigma\int_{\tau_0}^{\tau}\frac{d\tau}{\tau}$$

$$= \sigma\log_e\frac{\tau}{\tau_0}.$$

[For steam, the heat taken in at $\tau = \mathrm{L}$ (*i. e.*, in the change from the liquid to the gas).

Hence the change of entropy $= \frac{\mathrm{L}}{\tau}$]

Efficiency of an Engine working on the Rankine Cycle.

Example 6.—Find the efficiency of an engine working on the Rankine cycle; using the $\tau\phi$ diagram for the calculation.

Work done = area of ABCD (Fig. 106.)

= ABCK+ADMN−DKMN

$= \frac{q_1\mathrm{L}_1}{\tau_1}(\tau_1-\tau_2)$+heat taken in from τ_2 to $\tau_1-(\tau_2\times\mathrm{DK})$.

Now DK = the change in entropy from water at τ_2 to water at τ_1

$= \log_e\frac{\tau_1}{\tau_2}$ as proved above.

Hence the work done $= \frac{q_1 L_1}{\tau_1}(\tau_1-\tau_2)+(\tau_1-\tau_2)-\tau_2 \log \frac{\tau_1}{\tau_2}$

$= (\tau_1-\tau_2)\left(1+\frac{q_1 L_1}{\tau_1}\right)-\tau_2 \log \frac{\tau_1}{\tau_2}.$

The heat put in $= \tau_1-\tau_2+q_1 L_1$ (q_1 being the dryness fraction at τ_1),

and thus the efficiency $\eta = \dfrac{(\tau_1-\tau_2)\left(1+\frac{q_1 L_1}{\tau_1}\right)-\tau_2 \log \frac{\tau_1}{\tau_2}}{\tau_1-\tau_2+q_1 L_1}.$

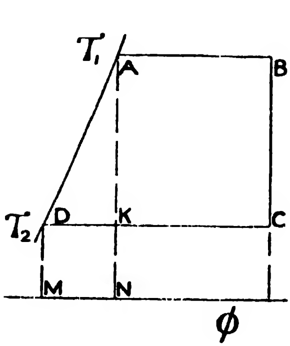


FIG. 106.

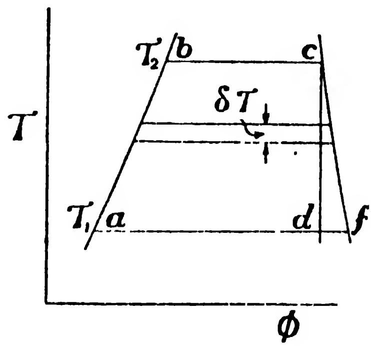


FIG. 107.

Efficiency of an Engine.

Efficiency of an Engine working on the Rankine Cycle, with steam kept saturated by jacket steam.

Example 7.—Find the efficiency of the engine whose cycle is given by *abcf* in Fig. 107.

Work done = area *abcf*

$$= \int_{\tau_1}^{\tau_2} \frac{L}{\tau} d\tau \quad \left(\text{the summation being of } \textit{horizontal strips}\right)$$

$$= \int_{\tau_1}^{\tau_2} \frac{a+b\tau}{\tau} d\tau \quad \left\{\begin{array}{r}\text{for } L = 1115-\cdot 7t \\ = 1437-\cdot 7\tau \\ = a+b\tau\end{array}\right\}$$

$$= \int_{\tau_1}^{\tau_2} \frac{a}{\tau} d\tau + \int_{\tau_1}^{\tau_2} b\,d\tau$$

$$= a \log_e \frac{\tau_2}{\tau_1} + b(\tau_2-\tau_1).$$

Total heat received $= L_2+\tau_2-\tau_1+H_f$ . . . (1)

total heat rejected $= L_1$ . . . . (2)

Hence the work done = (1) — (2).

$$\text{from which } H_f = a\log\frac{\tau_2}{\tau_1}+b(\tau_2-\tau_1)-(L_2-L_1)-(\tau_2-\tau_1)$$
$$= a\log\frac{\tau_2}{\tau_1}+b(\tau_2-\tau_1)-(a+b\tau_2-a-b\tau_1)-(\tau_2-\tau_1)$$
$$= a\log\frac{\tau_2}{\tau_1}-(\tau_2-\tau_1).$$

Now the total heat received $= L_2+\tau_2-\tau_1+H_f$

$$= L_2+\tau_2-\tau_1+a\log\frac{\tau_2}{\tau_1}-(\tau_2-\tau_1)$$
$$= a\log\frac{\tau_2}{\tau_1}+L_2.$$

Hence
$$\eta = \frac{a\log\frac{\tau_2}{\tau_1}+b(\tau_2-\tau_1)}{a\log\frac{\tau_2}{\tau_1}+a+b\tau_2}$$

where $a = 1437$ and $b = -\cdot 7$.

Example 8.—To prove that the equation for adiabatic expansion of air is $pv^\gamma = C$, where

$$\gamma = \frac{\text{specific heat at constant pressure}}{\text{specific heat at constant volume}} = \frac{K_p}{K_v}.$$

Dealing throughout with 1 lb. of air, let the air expand under constant pressure from conditions $p_1\ v_1\ \tau_1$ to $p_1\ v\ \tau$.

Then the heat added $= K_p(\tau-\tau_1) = K_p\left(\frac{p_1 v}{C_1}-\frac{p_1 v_1}{C_1}\right)$
$$= K_p p_1\left(\frac{v-v_1}{C_1}\right).$$

Now keep the volume constant at v, and subtract as much heat as was previously added: then the pressure falls to p_2 and the temperature to τ_2.

The heat subtracted $= K_v(\tau-\tau_2) = K_v\left(\frac{p_1 v}{C_1}-\frac{p_2 v}{C_1}\right)$
$$= \frac{K_v v}{C_1}(p_1-p_2).$$

Now, if the changes are regarded as being very small, we may write δv for $v-v_1$ and δp for p_1-p_2

and thus $\quad -K_v v\delta p = K_p p\delta v$

whence $\quad \int\frac{dp}{p} = -\frac{K_p}{K_v}\int\frac{dv}{v}$

$$\log p = -\gamma\log v + \log\text{ (constant)}$$

i. e., $\quad p = Cv^{-\gamma}$

or $\quad pv^\gamma = C.$

Examples relating to Loaded Beams.

Example 9.—Prove the most important rule

$$\frac{M}{I} = \frac{E}{R} = E\frac{d^2y}{dx^2}$$

applied to a loaded beam; M, I and E having their usual meanings, and R being the radius of curvature of the bent beam.

Assuming the beam to be originally straight, take a section of length l along the neutral lamina, and let $l+\delta l$ be the strained length at distance y (Fig. 108).

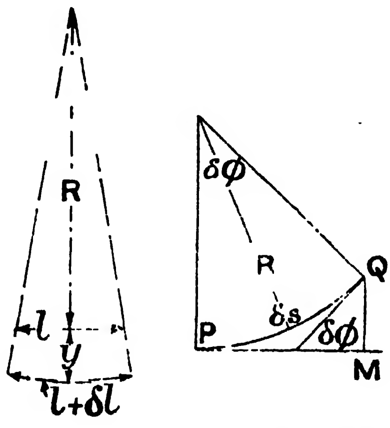


FIG 108.—Problem on Loaded Beam.

Then, if R = radius of curvature,

$$\frac{l+\delta l}{l} = \frac{R+y}{R}$$

whence $$\frac{l+\delta l}{l} - 1 = \frac{R+y}{R} - 1$$

or $$\frac{\delta l}{l} = \frac{y}{R}$$

but $$E = \frac{\text{stress}}{\text{strain}} = \frac{f}{\frac{\delta l}{l}} = \frac{f}{\frac{y}{R}} = \frac{fR}{y}$$

and thus $$\frac{E}{R} = \frac{f}{y}$$

but it has already been proved (see p.239) that $\frac{M}{I} = \frac{f}{y}$.

Hence $$\frac{M}{I} = \frac{E}{R}.$$

The total curvature of an arc of a curve is the angle through which the tangent turns as its point of contact moves from one end of the arc to the other; and the mean curvature is given by the total curvature divided by the length of arc.

In Fig. 108 $\delta\phi$ = total curvature for the arc δs, and the mean curvature $= \frac{\delta\phi}{\delta s}$.

We know that the slope of the tangent is given by $\frac{dy}{dx}$

$$\therefore \tan\phi = \frac{dy}{dx}.$$

Now $\quad \frac{d}{d\phi}\tan\phi = \sec^2\phi \quad$ and $\quad \frac{d\tan\phi}{ds} = \frac{d\tan\phi}{d\phi} \times \frac{d\phi}{ds}$

$$\sec^2\phi \cdot \frac{d\phi}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right)$$

$$\sec^2\phi \times \frac{1}{R} = \frac{d}{ds}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right) \times \frac{dx}{ds}$$

$$= \frac{d^2y}{dx^2} \times \frac{dx}{ds}.$$

Hence $\quad \frac{1}{R} = \frac{d^2y}{dx^2} \times \frac{dx}{ds} \times \cos^2\phi$

$$= \frac{d^2y}{dx^2} \times \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \times \frac{1}{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left\{\begin{aligned} \tan\phi &= \frac{dy}{dx} \\ \tan^2\phi &= \left(\frac{dy}{dx}\right)^2 \\ \sec^2\phi &= 1+\tan^2\phi \\ &= 1+\left(\frac{dy}{dx}\right)^2 \end{aligned}\right\}$$

$$= \frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}.$$

When, as for a beam, $\frac{dy}{dx}$ is very small, $\left(\frac{dy}{dx}\right)^2$ may be neglected in comparison with 1, and hence

$$\frac{1}{R} = \frac{d^2y}{dx^2}.$$

This result may be arrived at more briefly, but approximately, in the following manner:—

$$\delta\phi = \delta\tan\phi \text{ very nearly (when the angle is very small).}$$

Hence $\frac{\delta\phi}{\delta s} = \frac{\delta\tan\phi}{\delta s} = \frac{\delta}{\delta x}\tan\phi$ = rate of change of the tangent (for PM and PQ are sensibly alike).

Thus
$$\frac{d\phi}{ds} = \frac{d}{dx} \cdot \frac{dy}{dx} = \frac{d^2y}{dx^2}$$

and
$$\frac{1}{R} = \frac{d^2y}{dx^2}.$$

$\therefore$
$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R} = E\frac{d^2y}{dx^2}.$$

In the use of this rule there should be no difficulty in finding expressions for y in terms of x in cases in which the beam is simply supported; for an expression is found for the bending moment at distance x from one end, or the centre, whichever may be more convenient, and then the relation

$$\frac{M}{I} = E\frac{d^2y}{dx^2}$$

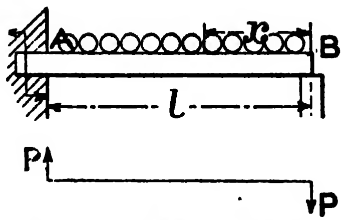


FIG. 109.—Beam Uniformly Loaded.

is used; whence double integration from the equation so formed gives an expression for the deflected form.

A few harder cases are here treated, the beam not being simply supported.

Example 10.—A beam is fixed at one end and supported at the other; the loading is uniform, w being the intensity. Find the equation of the deflected form.

We must first find the force P (part of the couple keeping the end fixed) and then combine this force with the reaction at B calculated on the assumption that the beam is simply supported. Referring to Fig. 109:—

If the beam is simply supported, the bending moment at distance x from B
$$= \frac{wl}{2} \cdot x - \frac{wx^2}{2}.$$

Hence the actual bending moment
$$= M = \frac{wlx}{2} - \frac{wx^2}{2} - Px$$

i. e.,
$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2} - Px$$

whence, by integration,
$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{Px^2}{2} + C_1$$

but $$\frac{dy}{dx}=0 \text{ when } x=l$$

for the deflected form is horizontal at this end.

$$\therefore \qquad 0=\frac{wl^3}{4}-\frac{wl^3}{6}-\frac{Pl^2}{2}+C_1$$

i. e., $$C_1=\frac{Pl^2}{2}-\frac{wl^3}{12}.$$

Hence $$EI\frac{dy}{dx}=\frac{wlx^2}{4}-\frac{wx^3}{6}-\frac{Px^2}{2}+\frac{Pl^2}{2}-\frac{wl^3}{12}.$$

Integrating, $$EIy=\frac{wlx^3}{12}-\frac{wx^4}{24}-\frac{Px^3}{6}+\frac{Pl^2x}{2}-\frac{wl^3x}{12}+C_2.$$

In this equation there are the two unknowns P and C_2, and to evaluate them we must form two equations from the statements

$$y=0 \text{ when } x=0 \quad \text{and} \quad y=0 \text{ when } x=l.$$

If $y=0$ when $x=0$, then it is readily seen that $C_2=0$.

Also if $x=l$ $$0=\frac{wl^4}{12}-\frac{wl^4}{24}-\frac{Pl^3}{6}+\frac{Pl^3}{2}-\frac{wl^4}{12}$$

whence $$P=\frac{wl}{8}.$$

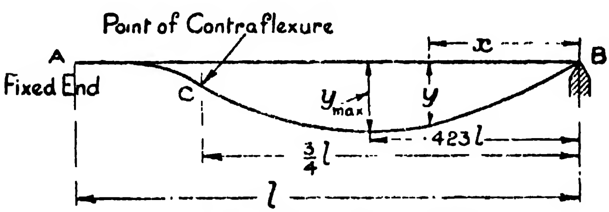


FIG. 110.—Deflected Form of Beam.

If the beam were simply supported, the upward reaction would be $\frac{wl}{2}$, and thus the net reaction $=\frac{wl}{2}-\frac{wl}{8}=\frac{3}{8}wl.$

Substituting $\frac{wl}{8}$ in place of P in the expression for y, we arrive at the equation of the deflected form

$$y=\frac{w}{48EI}(3lx^3-2x^4-l^3x),$$

the curve for which is shown in Fig. 110.

We may now proceed to find where the maximum deflection occurs, and also the position of the point of contraflexure.

$$\frac{dy}{dx}=\frac{w}{48EI}(9lx^2-8x^3-l^3)$$

and $$\frac{dy}{dx}=0 \text{ if } 9lx^2-8x^3=l^3,$$

the solution of which, applicable to the present case, is $x=\cdot4215l.$

The maximum deflection is thus

$$y_{\max} = \frac{wl^4}{48EI}(\cdot 228 - \cdot 064 - \cdot 423)$$

$$= -\frac{\cdot 0054 wl^4}{EI}.$$

To find C the point of inflexion or contraflexure

$$\frac{d^2y}{dx^2} = \frac{w}{48EI}(18lx - 24x^2)$$

and $$\frac{d^2y}{dx^2} = 0 \quad \text{if } x = 0 \quad \text{or if } x = \tfrac{3}{4}l.$$

Example 11.—A beam is fixed at one end and supported at the other, the loading and the section both varying. Find the equation of the deflected form.

Let m = bending moment at a point distant x from B if the beam were simply supported, and let P = the force of the fixing couple. (See Fig. 109.)

Then $$M = m - Px$$

i. e., $$EI\frac{d^2y}{dx^2} = m - Px$$

and $$E\frac{d^2y}{dx^2} = \frac{m}{I} - \frac{Px}{I}$$

: equation being written in this form since I is now a variable.

By integrating $$E\frac{dy}{dx} = \int_0^x \frac{m}{I}dx - P\int_0^x \frac{x}{I}dx + C_1 \quad \ldots \ldots \quad (1)$$

Now $$\frac{dy}{dx} = 0 \text{ when } x = l.$$

$$C_1 = P\int_0^l \frac{x}{I}dx - \int_0^l \frac{m}{I}dx,$$

i. e., C_1 can be found, for the two integrals may be evaluated.

By integrating (1),

$$Ey = \int_0^x\int_0^x \frac{m}{I}(dx)^2 - P\int_0^x\int_0^x \frac{x}{I}(dx)^2 + C_1x + C_2.$$

But $y = 0$ when $x = l$ and also when $x = 0$, and thus $C_2 = 0$

and $$0 = \int_0^l\int_0^x \frac{m}{I}(dx)^2 - P\int_0^l\int_0^x \frac{x}{I}(dx)^2 + l\left\{P\int_0^l \frac{x}{I}dx - \int_0^l \frac{m}{I}dx\right\},$$

i. e., P can be found.

The integrations must be performed graphically and with extreme care, or otherwise very serious errors arise.

Example 12.—A beam is fixed at both ends and the loading and the section both vary. Find the equation of the deflected form.

Let m_1 and m_2 be the end fixing couples; then to keep the system in equilibrium it is necessary to introduce equal and opposite forces P (Fig. 111), *i. e.*, $Pl+m_2=m_1$.

Let $m=$ the "simply supported" bending moment at section distant x from the right-hand end.

Then $$M=m-m_2-Px$$

and consequently $$EI\frac{d^2y}{dx^2}=m-m_2-Px$$

or $$E\frac{d^2y}{dx^2}=\frac{m}{I}-\frac{m_2}{I}-\frac{Px}{I}.$$

Integrating,

$$E\frac{dy}{dx}=\int_0^x\frac{m}{I}dx-m_2\int_0^x\frac{dx}{I}-P\int_0^x\frac{x}{I}dx+C_1.$$

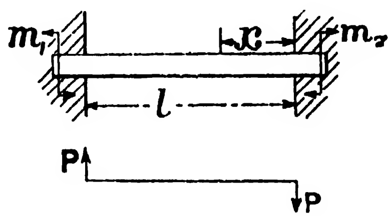


FIG. 111.

Now $$\frac{dy}{dx}=0 \text{ when } x=0 \text{ or } l$$

hence $$C_1=0 \quad (\text{taking } x=0)$$

and also, taking $x=l$,

$$0=\int_0^l\frac{m}{I}dx-m_2\int_0^l\frac{1}{I}dx-P\int_0^l\frac{x}{I}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating again,

$$Ey=\int_0^x\int_0^x\frac{m}{I}(dx)^2-m_2\int_0^x\int_0^x\frac{1}{I}(dx)^2-P\int_0^x\int_0^x\frac{x}{I}(dx)^2+C_2.$$

Now $y=0$ when $x=0$ and also when $x=l$

Then taking $x=0$, $C_2=0$,

and taking $x=l$,

$$0=\int_0^l\int_0^x\frac{m}{I}(dx)^2-m_2\int_0^l\int_0^x\frac{1}{I}(dx)^2-P\int_0^l\int_0^x\frac{x}{I}(dx)^2 \quad . \quad . \quad . \quad . \quad (2)$$

From equations (1) and (2) the values of m_2 and P (and hence m_1) may be found, the integration being graphical (except in a few special cases); and again it must be emphasised that the integration must be performed most accurately.

Example 13.—A uniform rectangular beam, fixed at its ends, is 20 ft. long, and has a load of 10 tons at its centre and one of 7 tons at 5 ft. from one end. Find the fixing couples and the true B.M. diagram.

This is a special case of *Example* 12 since the section, and therefore I, is constant.

The B.M. diagram for the beam if simply supported would be as ABCD (Fig. 112).

The bending moment diagram, due to the fixing couples only, would have the form of a trapezoid, as APQD.

Unless the integration, explained in the previous example, is done extremely carefully, there will be serious errors in the results; and since there are only the two loads to consider, it is rather easier to

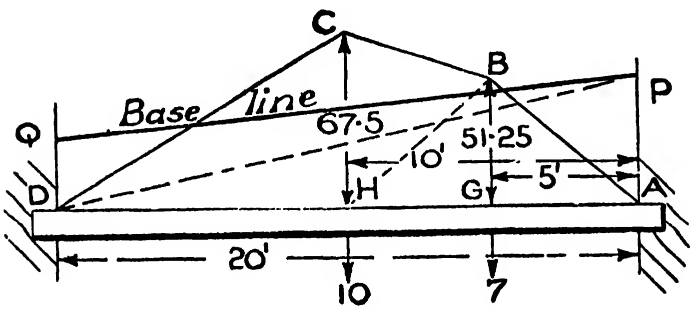


FIG. 112.—Fixing Couples and B.M. Diagram of Loaded Beam.

work according to the Goodman scheme. [See *Mechanics applied to Engineering*, by Goodman.]

According to this plan: (1) the opposing areas (*i. e.*, of the *free* and *fixing* bending moment diagrams) must be equal; and (2) the centroids of the opposing areas must be on the same vertical, *i. e.*, their centroid verticals must coincide.

To satisfy condition (1),

$$\text{Area of ABCD} = \left(\frac{1}{2}\times 5\times 51{\cdot}25\right)+\left(\frac{51{\cdot}25+67{\cdot}5}{2}\times 5\right)+\left(\frac{1}{2}\times 10\times 67{\cdot}5\right)$$

$$= 762{\cdot}5.$$

$$\text{Area of APQD} = \frac{1}{2}\times 20\times(m_1+m_2) \text{ where } \left.\begin{array}{r} m_1 = \text{AP} \\ \text{and } m_2 = \text{DQ}\end{array}\right\}$$

$$= 10\,(m_1+m_2).$$

Equating these areas, $m_1+m_2 = 76{\cdot}25$ . . . . (1)

To satisfy condition (2), taking moments about AP,

$$\text{Moment of ABG} = \frac{5}{2}\times 51\cdot 25\times\frac{2}{3}\times 5 \qquad = 427$$

$$\text{Moment of BGH} = \frac{5}{2}\times 51\cdot 25\times\left(5+\frac{1}{3}\times 5\right) = 854$$

$$\text{Moment of BHC} = \frac{5}{2}\times 67\cdot 5\times\left(5+\frac{2}{3}\times 5\right) = 1405$$

$$\text{Moment of DCH} = \frac{10}{2}\times 67\cdot 5\times\left(10+\frac{1}{3}\times 10\right) = 4500$$

i. e., total moment of ABCD about AP $= 7186$

$$\text{Moment of APD} = \frac{20}{2}\times m_1\times\frac{20}{3} \qquad = \frac{200}{3}m_1$$

$$\text{Moment of DPQ} = \frac{20}{2}\times m_2\times\frac{40}{3} \qquad = \frac{400}{3}m_2$$

Hence $$\frac{200}{3}m_1+\frac{400}{3}m_2 = 7186 \quad . \quad . \quad . \quad . \quad (2)$$

The solution of equations (1) and (2) for m_1 and m_2 gives the results

$$m_1 = 44\cdot 71 \text{ and } m_2 = 31\cdot 54$$

Thus PQ is the true base of the complete bending moment diagram, AP being made equal to 44·71, and DQ equal to 31·54.

Shearing Stress in Beams.

Example 14.—To find an expression for the maximum intensity of shearing stress over a beam section.

The shearing stress at any point in a vertical section of a beam is always accompanied by shearing stress of equal intensity in a horizontal plane through that point.

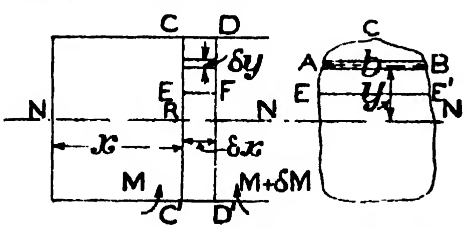


FIG. 113.—Shearing Stress in Beams.

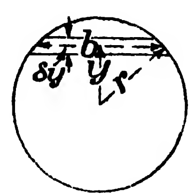


FIG. 114.

We require to know the tangential or shearing stress f at E on the plane CEC′ (Fig. 113); this must be equal to the tangential stress in the direction EF on the plane EF at right angles to the paper. Suppose that the bending moment at CC′=M and that at DD′=M+δM. Then the total pushing forces on DF > total pushing forces on CE, the difference being the tangential forces on EFE′.

Let P = the total pushing force on ECE′

then
$$P = \int_{RE}^{RC} (\text{stress}) \times \text{area} \qquad \left\{\begin{array}{l} \frac{\text{stress}}{y} = \frac{M}{I} \\ \text{area} = b\,dy \end{array}\right\}$$
$$= \int_{RE}^{RC} \frac{yM}{I} b\,dy$$
$$= \frac{M}{I}\int_{RE}^{RC} by\,dy = \frac{M}{I} \times \text{1st moment of area ECE}'.$$

Now the tangential force on EFE′ = stress × area
$$= f \times EE' \times \delta x$$

and this must equal the difference in the total pushing forces on DF and CE, *i. e.*, δP.

Hence
$$\delta P = f \times EE' \times \delta x$$

i. e.,
$$\frac{\delta M}{I} \times \text{1st moment of area ECE}' = f \times EE' \times \delta x,$$

but
$$\frac{\delta M}{\delta x} = \text{rate of change of B.M.} = \text{shear} = F \text{ (say)}.$$

Hence the maximum intensity of shearing stress f
$$= \text{1st moment of area ECE}' \times \frac{F}{I} \times \frac{1}{EE'}$$

or, as it is usually written,
$$f = \frac{FS\bar{y}}{bI}$$

where S = an area such as CEE′ and $\bar{y}$ = distance of its centroid from the neutral axis.

Example 15.—Find the maximum intensity of shearing stress, when the section is circular, of radius r (see Fig. 114).

For this section
$$I = \frac{\pi}{4}r^4.$$

Applying the rule proved above:—

the maximum intensity
$$= \frac{F}{2rI}\int_0^r by\,dy \qquad \text{since } 2r \text{ corresponds to EE}' \text{ in } \textit{Example } 14.$$
$$= \frac{F}{2rI}\int_0^r 2(r^2-y^2)^{\frac{1}{2}} y\,dy$$
$$= \frac{F}{rI}\int_0^r u^{\frac{1}{2}} \frac{du}{-2} \qquad \left\{\begin{array}{l} \text{where } u = r^2 - y^2 \\ \frac{du}{dy} = -2y \end{array}\right\}$$
$$= -\frac{F}{2rI}\left(\frac{2}{3}(r^2-y^2)^{\frac{3}{2}}\right)_0^r$$
$$= \frac{F \times 4}{2r\pi r^4} \times \frac{2}{3}r^3 = \frac{4}{3} \cdot \frac{F}{\pi r^2}$$
$$= \frac{4}{3} \times \text{mean intensity}.$$

Example 16.—A uniformly tapered cantilever of circular cross section is built in at one end and is loaded at the other. The diam. at the loaded end is D ins. and the taper is t ins. per in. of length. Find an expression for the distance of the most highly stressed section from the free end of the beam due to bending moment only. Neglect the weight of the cantilever.

Let l be the length in ins. and W the load at the free end. Consider a section distant x ins. from the free end; then the diam. here is $D-tx$, and the bending moment Wx.

Also the value of I for the section considered is

$$\frac{\pi}{64}(\text{diam.})^4 = \frac{\pi}{64}(D-tx)^4.$$

Hence the skin stress $f = \frac{My}{I} = Wx\left(\frac{D-tx}{2}\right)\times\frac{64}{\pi(D-tx)^4}$

$$= \frac{32xW}{\pi(D-tx)^3}$$

$$= \frac{K}{\frac{D^3-3D^2tx+3Dt^2x^2-t^3x^3}{x}} \text{ where } K=\frac{32W}{\pi} \text{ (a constant),}$$

and f is a maximum when the denominator is a minimum.

Let $\qquad N = \text{denominator}$

$$= D^3x^{-1} - 3D^2t + 3Dt^2x - t^3x^2$$

then $\qquad \frac{dN}{dx} = -\frac{D^3}{x^2} + 3Dt^2 - 2t^3x$

and $\qquad \frac{dN}{dx} = 0$ when $2t^3x^3 - 3Dt^2x^2 + D^3 = 0$

i. e., when

$$2t^3x^3 - 2Dt^2x^2 - Dt^2x^2 + D^3 = 0$$

$$2t^2x^2(tx-D) - D(t^2x^2 - D^2) = 0$$

$$(tx-D)(2t^2x^2 - Dtx - D^2) = 0$$

$$(tx-D)(2tx+D)(tx-D) = 0$$

i. e., when $\qquad x = \frac{D}{t}$ or $-\frac{D}{2t}$.

Thus the stress is maximum at a section distant $\underline{\frac{D}{t} \text{ ins. from the}}$ free end.

Example 17.—To find the deflection of the muzzle of a gun.

This is an instructive example on the determination of the deflection of a cantilever whose section varies.

The muzzle is divided into a number of elementary discs, the volumes of these found (and hence the weights) so that the curve of loads can be plotted. Integration of this curve gives the curve of shear, and integration of the curve of shear gives the B.M. diagram.

The values of I must next be calculated for each disc, and a new curve plotted with ordinates equal to $\frac{\text{B.M.}}{\text{I}}$: then double integration of this gives the deflected form.

It is necessary to use the ordinates of this curve to find, first the time of the fundamental oscillation and thence the upward velocity due to the deflection.

Call the deflection at any section y, and the load or weight of the small disc w.

Find the sum of all products like wy^2 and also find the sum of all products like wy.

(Suitable tabulation will facilitate matters.)

Then T (time of oscillation)

$$= 2\pi\sqrt{\frac{\Sigma wy^2}{g\Sigma wy}}.$$

If $Y =$ maximum deflection, assuming the motion to be S.H.M., then the upward velocity v is obtained from

$$vT = 2\pi Y$$

or

$$v = \frac{2\pi Y}{T}.$$

Examples on Applied Electricity.

Example 18.—Arrange n electric cells partly in series and partly in parallel to obtain the maximum current from them through an external resistance R. (Let the internal resistance of each cell $= r$, and let the E.M.F. of each cell $= v$.)

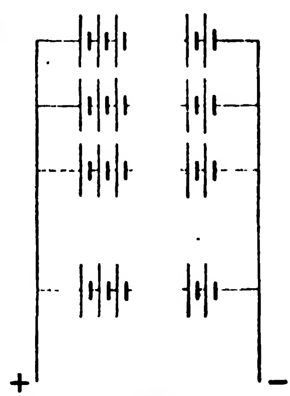


FIG. 115.—Maximum Current from Electric Cells.

Suppose the mixed circuit is as shown in Fig. 115, *i. e.*, with x cells per row and therefore $\frac{n}{x}$ rows.

Then the total E.M.F. of 1 row $= xv$ and total internal resistance of 1 row $= xr$, but as there are $\frac{n}{x}$ rows, the total internal resistance is $\frac{1}{\frac{n}{x}}$ that of 1 row, *i. e.*, the total internal resistance $= \frac{rx^2}{n}$; but the E.M.F. is unaltered: in reality the effect being that of one large cell, the area being greater and thus the resistance less.

Hence the current $I = \dfrac{\text{total E.M.F.}}{\text{total resistance}} = \dfrac{xv}{\dfrac{rx^2}{n}+R}$

$$= \frac{v}{\dfrac{rx}{n}+\dfrac{R}{x}}$$

and I is maximum when the denominator is a minimum.

Let D = the denominator, then

$$\frac{dD}{dx} = \frac{r}{n} - \frac{R}{x^2} \text{ and } \frac{dD}{dx} = 0 \quad \text{when} \quad R = \frac{rx^2}{n},$$

i. e., external = internal resistance.

Example 19.—To find an expression for the time of discharge of an electric capacitor of capacitance K, discharging through a constant resistance R.

Let v = potential difference between the coatings at any time t.

Then, by Ohm's law, the current $I = \dfrac{v}{R}$.

But the current is given by the rate of diminution of the quantity q and $q = Kv$.

Hence $$I = -\frac{dq}{dt} = -\frac{d\,Kv}{dt} = -K\frac{dv}{dt}$$

and thus $$\frac{v}{R} = -K\frac{dv}{dt}.$$

If V_1 = the difference of potential at the start, *i. e.*, at $t = 0$, and V_2 = the difference of potential at the end of T secs,

Separating the variables and integrating,

$$-\int_{V_1}^{V_2}\frac{dv}{v} = \frac{1}{KR}\int_0^T dt$$

whence $$\log\frac{V_1}{V_2} = \frac{T}{KR}$$

or $$\frac{V_1}{V_2} = e^{\frac{T}{KR}}$$

i. e., $$\frac{V_2}{V_1} = e^{-\frac{T}{KR}}$$

or the time taken to lower the voltage from

$$V_1 \text{ to } V_2 = KR\log\frac{V_1}{V_2}$$

Example 20.—If R = the electric resistance of a circuit, L = its self-inductance, I = the current flowing and V = the voltage, then

$$V = RI + L\frac{dI}{dt}.$$

Solve this equation for the cases when $V = 0$, $I = I_0 \sin \omega t$ and $V = V_0 \sin \omega t$.

For the case of a steady current $V = RI$ since L is zero, and this corresponds to the equation of uniform motion in mechanics, whilst the equation $V = RI + L\frac{dI}{dt}$ may be compared with that for accelerated motion, V being the force. Thus the second term of the equation may be regarded as one expressing the " inertia " or " reluctance to change," and since the current may vary according to various laws, the rate of change $\frac{dI}{dt}$ can have a variety of values.

Dealing with the cases suggested :—

(1) If $V = 0$, then
$$L\frac{dI}{dt} + RI = 0$$
$$\left(D + \frac{R}{L}\right)I = 0$$

Thus $D = -\frac{R}{L}$ and $\underline{I = Ae^{-\frac{Rt}{L}}}.$

(2) If
$$I = I_0 \sin \omega t$$
$$\frac{dI}{dt} = I_0\omega \cos \omega t$$
$$V = LI_0\omega \cos \omega t + RI_0 \sin \omega t$$
$$= \underline{I_0\sqrt{R^2 + L^2\omega^2} \sin \left(\omega t + \tan^{-1} \frac{L\omega}{R}\right)}.$$

(3) If
$$V = V_0 \sin \omega t$$
$$L\frac{dI}{dt} + RI = V_0 \sin \omega t$$
$$\left(D + \frac{R}{L}\right)I = \frac{V_0}{L} \sin \omega t.$$

The complementary function is $I_c = Ae^{-\frac{Rt}{L}}$. The particular integral
$$I_p = \frac{V_0}{L\left(D + \frac{R}{L}\right)} \sin \omega t$$
$$= \frac{V_0(LD - R) \sin \omega t}{L^2D^2 - R^2}$$
$$= \frac{V_0(L\omega \cos \omega t - R \sin \omega t)}{-L^2\omega^2 - R^2}$$
$$= \frac{V_0(R \sin \omega t - L\omega \cos \omega t)}{R^2 + L^2\omega^2}$$

and
$$I = I_c + I_p$$
$$= \underline{Ae^{-\frac{Rt}{L}} + \frac{V_0(R \sin \omega t - L\omega \cos \omega t)}{R^2 + L^2\omega^2}}.$$

Example 21.—To find expressions for the potential and the current at any points along a long uniform conductor.

At a distance x from the "home" end let the steady potential to the ground $=$ E and the steady current be i.

Let the resistance of 1 unit of length of the conductor $= r$ and let the leakage of 1 unit of length of the insulation $= l$.

Consider a small length of conductor δx.

Its resistance $= r\delta x$ and the leakage $= l\delta x$.

$$\text{Hence the drop in potential} = i \times r\delta x \qquad (1)$$

$$\text{and the drop in the current} = E \times l\delta x \qquad (2)$$

i. e., from (1) $\quad \delta E = -ir\delta x$

$$\text{or} \qquad \frac{\delta E}{\delta x} = -ir \qquad (3)$$

and from (2) $\quad \delta i = -El\delta x$

$$\text{or} \qquad \frac{\delta i}{\delta x} = -El \qquad (4)$$

Writing (3) and (4) in their limiting forms

$$\frac{dE}{dx} = -ir, \qquad \frac{di}{dx} = -El.$$

$$\text{Differentiating,} \quad \frac{d^2E}{dx^2} = \frac{d}{dx}(-ir) = -r\frac{di}{dx} = rEl \qquad (5)$$

$$\text{and} \quad \frac{d^2i}{dx^2} = \frac{d}{dx}(-El) = -l\frac{dE}{dx} = ril \qquad (6)$$

To solve these equations, let $D = \dfrac{d}{dx}$

then $\quad D^2E = rlE$

i. e., $\quad D^2 = rl$

$$D = \pm\sqrt{rl}.$$

$$\therefore \qquad \frac{dE}{dx} = \sqrt{rl}E \text{ or } \frac{dE}{dx} = -\sqrt{rl}E.$$

Separating the variables,

$$\frac{dE}{E} = \sqrt{rl}dx \text{ or } \frac{dE}{E} = -\sqrt{rl}dx.$$

Integrating,

$$\log E = \sqrt{rl}x + C_1 \text{ or } \log E = -\sqrt{rl}x + C_2$$

i. e., $\quad E = A_1e^{\sqrt{rl}x} + A_2e^{-\sqrt{rl}x}$

or, if the constants are suitably chosen,

$$E = A\cosh\sqrt{rl}x + B\sinh\sqrt{rl}x.$$

In like manner, $i = \text{C} \cosh \sqrt{rl}\, x + \text{D} \sinh \sqrt{rl}\, x.$

When $x = \text{L}$

$$\text{E} = \text{A} \cosh \sqrt{rl}\,\text{L} + \text{B} \sinh \sqrt{rl}\,\text{L}.$$

When $x = 0$

$$\text{E} = \text{A}$$

and hence the constants can be found.

Examples on Strengths of Materials.

Example 22.—To find the shape assumed by a chain loaded with its own weight only; the weight per foot being w. To find also expressions for the length of arc and the tension at any point.

Let $s =$ the length of the arc AB (Fig. 116): then the weight of this portion $= ws$.

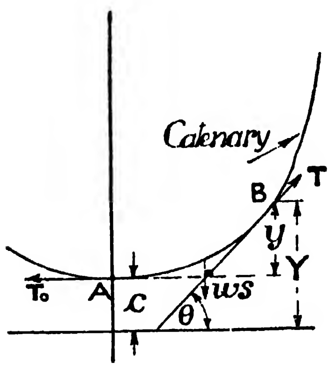


FIG. 116.

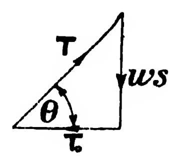


FIG. 117.

Draw the triangle of forces for the three forces T, T_0 and ws (Fig. 117).

Let it be assumed that T_0 (the horizontal tension) $= wc$, where c is some constant.

Then, from Figs. 116 and 117

$$\frac{dy}{dx} = \tan\theta = \frac{ws}{\text{T}_0} = \frac{ws}{wc} = \frac{s}{c}.$$

Now, as proved on p. 201,

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{c^2}{s^2}} = \frac{\sqrt{s^2+c^2}}{s}$$

$$\text{or} \quad \frac{s\,ds}{\sqrt{c^2+s^2}} = dy \quad \ldots\ldots\ldots \quad (1)$$

To integrate the left-hand side, let $u = c^2+s^2$

then $$\frac{du}{ds} = 2s$$

and $$\int\frac{s\,ds}{\sqrt{c^2+s^2}} = \int\frac{s\,du}{2s\,u^{\frac{1}{2}}} = u^{\frac{1}{2}} = \sqrt{c^2+s^2}.$$

Thus, by integration of equation (1),

$$\sqrt{c^2+s^2} = y + C_1.$$

Now at the point A (Fig. 116) $s = 0$ and $y = 0$

hence $$\sqrt{c^2} = C_1 \quad \text{or} \quad C_1 = c.$$

Thus $$\sqrt{c^2+s^2} = y + c.$$

Squaring $$c^2+s^2 = y^2+c^2+2yc$$

or $$s^2 = y^2+2yc \quad \text{and} \quad s = \sqrt{y^2+2yc}$$

but as proved above $\frac{s}{c} = \frac{dy}{dx}$ or $s = c\frac{dy}{dx}$

hence $$c\frac{dy}{dx} = \sqrt{y^2+2yc}.$$

Separating the variables

$$\frac{dy}{\sqrt{y^2+2yc}} = \frac{dx}{c}$$

i. e., $$\frac{dy}{\sqrt{y^2+2yc+c^2-c^2}} = \frac{dx}{c} \quad \text{or} \quad \frac{dy}{\sqrt{(y+c)^2-c^2}} = \frac{dx}{c}.$$

Integrating $$\int\frac{dy}{\sqrt{(y+c)^2-c^2}} = \int\frac{dx}{c}$$

and this integral is of the type discussed on p. 151; the result being

$$\log\left(\frac{y+c+\sqrt{y^2+2yc}}{c}\right) = \frac{x}{c}+C_2.$$

Now $x = 0$ when $y = 0$, x being measured from the vertical axis through A, and thus $\log\left(\frac{c}{c}\right) = C_2$ or $C_2 = 0$.

Thus $$\frac{x}{c} = \log\left(\frac{y+c+\sqrt{y^2+2yc}}{c}\right)$$

or in the exponential form

$$ce^{\frac{x}{c}} = y+c+\sqrt{y^2+2yc}.$$

Isolating the surd $$ce^{\frac{x}{c}}-(y+c) = \sqrt{y^2+2yc}.$$

Squaring $$c^2e^{\frac{2x}{c}} +y^2+c^2+2yc-2(y+c)\,ce^{\frac{x}{c}} = y^2+2yc$$

or $$c^2e^{\frac{2x}{c}} -2ce^{\frac{x}{c}}(y+c)+c^2 = 0.$$

Dividing through by $ce^{\frac{x}{c}}$

$$ce^{\frac{x}{c}}+ce^{-\frac{x}{c}}=2(y+c),$$

i.e.,
$$(y+c)=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right).$$

If now the axis of x be shifted downwards a distance c, then the new ordinate $Y=y+c$ and $Y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)=c\cosh\frac{x}{c}$.

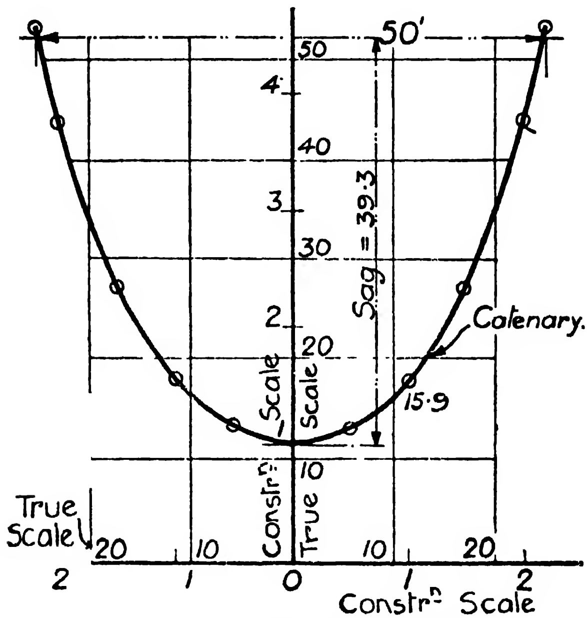


FIG. 118.—Catenary Form of a Cable.

Again, since $Y=y+c$
$$\frac{dY}{dx}=\frac{d(y+c)}{dx}=\frac{dy}{dx}$$

and also
$$\frac{d}{dx}c\cosh\frac{x}{c}=\frac{c}{c}\sinh\frac{x}{c}=\sinh\frac{x}{c}.$$

Then
$$\frac{dy}{dx}=\sinh\frac{x}{c}$$

but it has already been proved that

$$\frac{dy}{dx}=\frac{s}{c}$$

hence
$$\frac{s}{c}=\sinh\frac{x}{c}\quad\text{or}\quad s=c\sinh\frac{x}{c}.$$

To find the tension T at any point

$$T^2 = w^2s^2 + w^2c^2 \quad \text{from Fig. 117}$$
$$= w^2(s^2+c^2)$$
$$= w^2(c+y)^2 = w^2Y^2$$

or $$T = wY.$$

Thus the form taken by the chain is that for which the equation is $Y = c\cosh\left(\frac{x}{c}\right)$, -the equation of the catenary: the length of arc is given by $s = c\sinh\frac{x}{c}$, and the tension at any point is measured by the product of the ordinate at that point and the weight per foot of the chain.

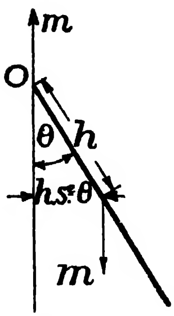


FIG. 119.

Fig. 118 shows the catenary for a cable weighing 3·5 lbs. per foot and strained to a tension of 40 lbs. weight, and the method of calculation for the construction of this curve is explained on p. 358 of Part I.

The tension at 10 ft. from the centre $= 3{\cdot}5 \times 15{\cdot}9 = 55{\cdot}6$ lbs. weight, since the ordinate there is 15·9.

Example 23.—To find the time of oscillation of a compound pendulum swinging through small arcs.

Let I be the moment of inertia of the pendulum about an axis through the point of suspension O (Fig. 119), and let h = the distance of the C. of G. from the point of suspension.

Then the couple acting, to produce the angular acceleration, = moment of inertia × angular acceleration.

[Compare the rule for linear motion, Force = mass × acceleration.]

Now the angular velocity $= \dfrac{d\theta}{dt}$

and hence the angular acceleration

$$= \frac{d^2\theta}{dt^2}.$$

Thus the couple acting $= I\dfrac{d^2\theta}{dt^2}$

and this couple is opposed by one whose arm is $h\sin\theta$, as is seen in the figure.

Thus $$I\frac{d^2\theta}{dt^2} = -mh\sin\theta = -mh\theta$$

since θ is supposed to be small, and consequently $\sin\theta = \theta$

or $$\frac{d^2\theta}{dt^2} = -\frac{mh}{I}\theta = -\omega^2\theta \quad \text{if} \quad \omega^2 = \frac{mh}{I}$$

and $$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0.$$

This equation is of the type dealt with in *Case* (3), p. 283, and the solution is $\theta = A\sin(\omega t + B)$.

The period of this function is $\frac{2\pi}{\omega}$; also the couple for angular displacement $\theta = mh\theta$; hence the couple for unit angular displacement (denoted by μ) $= mh$.

Hence $$t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mh}} = 2\pi\sqrt{\frac{I}{\mu}}$$

or $$t = 2\pi\sqrt{\frac{I}{\mu g}} \text{ if engineers' units are used.}$$

This might be written in the easily remembered form,

$$t = 2\pi\sqrt{\frac{I}{\text{unit moment of inertia per unit twist}}}.$$

If this formula is to be used in the determination of the modulus of rigidity of a sample of wire by means of torsional oscillations, h must be replaced by l, the length of the wire.

If f = skin stress, T = torque, and C = modulus of rigidity, d = dia. of wire; then

$$\theta = \frac{2fl}{Cd} \quad \text{and} \quad T = \frac{\pi}{16}fd^3$$

whence $$\theta = \frac{32Tl}{\pi d^4 C}$$

but $$\mu = \frac{T}{\theta} = \frac{\pi d^4 C}{32l}$$

and $$t = 2\pi\sqrt{\frac{I}{\mu g}}$$

$$= 2\pi\sqrt{\frac{32lI}{\pi d^4 Cg}}$$

hence $C = \frac{128\pi lI}{gd^4t^2}$ and thus C can be determined.

As regards the units, if l is in feet, I is in lbs. ft.², t in secs., and d in feet,

then $$C = \frac{\text{feet} \times \text{lbs.} \times \text{ft.}^2 \times \text{sec.}^2}{\text{feet} \times \text{ft.}^4 \times \text{sec.}^2}$$

$$= \frac{\text{lbs.}}{\text{ft.}^2}, \text{ i. e., C is in lbs. per sq. foot.}$$

If I is in lbs. ins.² and d is in ins., then C will be in lbs. per sq. in.

Example 24.—To find formulæ giving the radial and hoop stresses in thick cylinders subjected to internal stress.

We may attack this problem by either of two methods:—

Method 1.—In (*a*) Fig. 120 let the outside radius $= r_1$ and the inside radius $= r_0$; also let the internal pressure be p, and the crushing stress at right angles to the radii, or the hoop stress, $= q$.

It is rather easier to consider the stress on the outside to be greater than that on the inside: thus for an annulus of radius r and thickness δr, we take the internal stress as p and the external stress as $p+\delta p$.

Considering the element RS of the annule (subtending an angle of $\delta\theta$ at the centre), and dealing with the radial forces,

$$\begin{aligned}\text{Total radial force} &= (p+\delta p)\times\text{outer arc}-p\times\text{inner arc}\\ &= (p+\delta p)\times(r+\delta r)\delta\theta-pr\delta\theta\\ &= (pr+p\delta r+r\delta p+\delta p.\delta r-pr)\,\delta\theta\\ &= (p\,\delta r+r\,\delta p+\delta p.\delta r)\,\delta\theta\end{aligned}$$

(for a unit length of the cylinder).

This is balanced by two forces each $q\,\delta r.\delta\theta$, for

$$\frac{\frac{x}{2}}{q\delta r}=\sin\frac{\delta\theta}{2}=\frac{\delta\theta}{2}\text{ nearly}\qquad[(b)\text{ Fig. 120}]$$

i. e., $\quad x=q\,\delta r.\delta\theta$

x being the radial force.

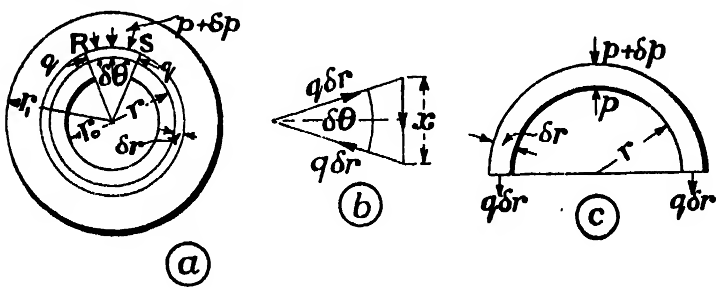


FIG. 120.—Stresses in Thick Cylinders.

Thus $\quad (p\,\delta r+r\delta p+\delta p.\delta r)\delta\theta=q\,\delta r.\delta\theta$

or, when δr is very small,

$$p\,dr+r\,dp=q\,dr.$$

Assume each longitudinal fibre to lengthen the same amount due to the secondary strains.

Then if $\sigma=$ Poisson's ratio and $\mathrm{E}=$ Young's modulus for the material,

$$\text{the extension due to } p \text{ will be } \frac{p}{\mathrm{E}}\times\frac{1}{\sigma}$$

$$\text{and the extension due to } q \text{ will be } \frac{q}{\mathrm{E}}\times\frac{1}{\sigma}$$

then, since the total extension is to be constant,

$$(p+q)\frac{1}{\sigma\mathrm{E}}=\text{constant,}$$

i. e., $\quad p+q=2\mathrm{A}$, say, for σ and E are constants.

Hence $\qquad r\,dp + p\,dr = q\,dr$

$$= (2A - p)dr$$

i. e., $\qquad r\,dp = 2(A-p)dr.$

Separating the variables and integrating,

$$\int \frac{dp}{2(A-p)} = \int \frac{dr}{r}$$

i. e., $\qquad -\tfrac{1}{2}\log(A-p) + \log C = \log r$

$$\text{or } r = \frac{C}{(A-p)^{\frac{1}{2}}}$$

i. e., $\qquad (A-p)^{\frac{1}{2}} = \frac{C}{r}$

$$A - p = \frac{C^2}{r^2}$$

$$\left.\begin{aligned} \text{or } p &= A + \frac{B}{r^2} \\ \text{but} \qquad p + q &= 2A \\ \text{and hence} \qquad q &= A - \frac{B}{r^2} \end{aligned}\right\}$$

The constants A and B are found from the conditions stated in any example.

Method 2.—According to this scheme q is taken as a tensile stress.

By the thin cylinder theory;—consider the equilibrium of the half elementary ring of unit length [(*c*) Fig. 120].

Then $\qquad (p \times 2r) - (p + \delta p)2(r + \delta r) = 2q\delta r$

whence $\qquad q\,dr = -p\,dr - r\,dp.$

From this point the work is as before except that 2A is written for $p - q$ and not for $p+q$ as in *Method* 1.

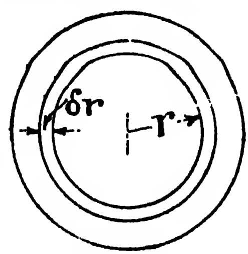


FIG. 121.

Example 25.—To find expressions for the stresses in Thick Spherical Shells.

Let $\qquad p =$ the radial pressure, $q =$ the hoop tension.

Take an elementary shell at radius r, the thickness being δr (Fig. 121).

Then $\qquad \pi r^2 p - \pi(r+\delta r)^2(p + \delta p) = 2\pi r\,q\delta r.$

$$\pi r^2 p - \pi r^2 p - \pi p(\delta r)^2 - 2\pi r\delta r p - \pi r^2\delta p - \pi\,\delta p(\delta r)^2 - 2\pi r\,\delta r.\delta p = 2\pi r\delta r q.$$

When δr is very small this equation reduces to

$$-2p\delta r - r\,\delta p = 2q\delta r$$

and hence $\qquad 2q = -2p - r\frac{dp}{dr}.$

Assuming the volumetric strain to be the same everywhere,

$$2l_y+l_z=C$$

where $l_y=$ the circumferential strain, and thus $2l_y=$ the superficial strain, and $l_z=$ the radial strain,

then it follows that

$$\frac{2q}{\sigma E}-\frac{p}{\sigma E}=C$$

i. e., $$2q-p=\text{Constant}=3A \text{ (say)}.$$

Now $$2q=-2p-r\frac{dp}{dr}$$

$\therefore$ $$3A+p=-2p-r\frac{dp}{dr}$$

i. e., $$3(A+p)=-\frac{rdp}{dr}.$$

Separating the variables and integrating,

$$\int-\frac{dr}{r}=\int\frac{dp}{3(A+p)}$$

i. e., $$-\log r=\frac{1}{3}\log(A+p)+\log C_1$$

whence $$(A+p)^{\frac{1}{3}}=\frac{1}{C_1r}$$

or $$A+p=\frac{1}{C_1{}^3r^3}$$

i. e., $$p=\frac{1}{C_1{}^3r^3}-A=\frac{2B}{r^3}-A \text{ (say)}.$$

Also $$2q=3A+p$$

$$=3A+\frac{2B}{r^3}-A=2A+\frac{2B}{r^3}$$

i. e., $$q=\frac{B}{r^3}+A.$$

Euler's Formulæ for Loaded Struts.

Example 26.—To obtain a formula giving the buckling load for a strut of length L and moment of inertia I.

Applying the ordinary rule

$$\frac{M}{I}=E\frac{d^2y}{dx^2}$$

Bending moment at $Q = M = -Py$ [(*a*) Fig. 122]

$\therefore$ $$IE\frac{d^2y}{dx^2} = -Py$$

i. e., $$\frac{d^2y}{dx^2} = -\frac{P}{IE}y.$$

Let $$\frac{P}{IE} = \omega^2$$

then $$\frac{d^2y}{dx^2} + \omega^2 y = 0$$

and the solution of this equation is, according to *Case* (3), p. 283,

$$y = A\sin(\omega x + B).$$

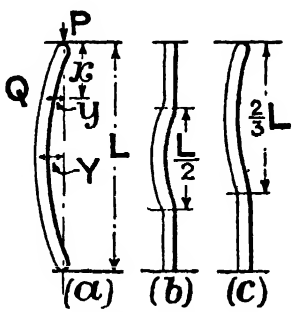


FIG. 122.

The various conditions of end fixing give rise to the following solutions:—

Case of ends rounded.—When $x = 0$, $y = 0$

then $0 = A\sin(0+B)$ and A is not zero

so that $B = 0$.

When $x = \frac{L}{2}$, $y = Y$ [(*a*) Fig. 122]

i. e., $$Y = A\sin\frac{\omega L}{2}$$

Obviously Y is the amplitude, *i. e.*, $A = Y$

or $$1 = \sin\frac{\omega L}{2}.$$

Thus we may write $\sin\frac{\pi}{2} = \sin\frac{\omega L}{2}$ $\left(\frac{\pi}{2}\text{ being the simplest angle having its sine} = 1\right)$

whence $$\omega L = \pi$$

and $$\sqrt{\frac{P}{IE}} \times L = \pi$$

or $$P = \frac{\pi^2 IE}{L^2}.$$

Case of both ends fixed.—The form taken by the column is as at (*b*) Fig. 122. The half-period of the curve is evidently $\frac{L}{2}$ in this case.

$$\text{But the period} = \frac{2\pi}{\omega}.$$

$$\therefore \qquad L = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{P}{IE}}}$$

whence
$$P = \frac{4\pi^2 IE}{L^2}.$$

Case of one end fixed.—The form taken by the column is as at (*c*) Fig. 122. The half-period in this case is $\frac{2}{3}L$, but, as before proved, the period is given by $\frac{2\pi}{\omega}$.

Hence
$$\frac{4}{3}L = \frac{2\pi}{\omega}$$

or
$$L = \frac{3\pi}{2\omega}.$$

Now
$$\omega = \sqrt{\frac{P}{IE}}$$

$$\therefore \qquad L^2 = \frac{9\pi^2 IE}{4P}$$

or
$$P = \frac{9\pi^2 IE}{4L^2}.$$

Tension in Belt passing round a Pulley.

Example 27.—To compare the tensions T_1 and T_2 at the ends of a belt passing round a pulley; the coefficient of friction between the belt and pulley being μ, and the angle of lap being θ radians.

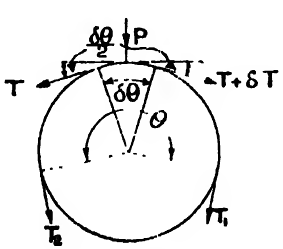


Fig. 123.

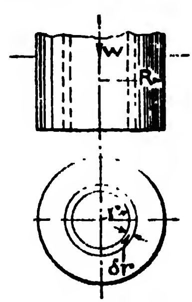


Fig. 124.

Consider a small element of belt subtending an angle of $\delta\theta$ at the centre of the pulley (see Fig. 123): then the tensions at the ends are respectively $T+\delta T$ and T.

Resolving the forces horizontally

$$(T+\delta T)\cos\frac{\delta\theta}{2} - T\cos\frac{\delta\theta}{2} = \mu P$$

i. e., $$\delta T\cos\frac{\delta\theta}{2} = \mu P$$

or in the limit $$dT = \mu P \quad \text{. . . . . (1)}$$

$$\text{for } \cos\frac{\delta\theta}{2} \to \cos 0, \text{ i. e., } 1.$$

Resolving the forces vertically,

$$P = (T+\delta T+T)\sin\frac{\delta\theta}{2}$$

$$= 2T\sin\frac{\delta\theta}{2} + \delta T\sin\frac{\delta\theta}{2}$$

$$= 2T\frac{\delta\theta}{2} + \delta T\frac{\delta\theta}{2} \quad \left(\text{for } \sin\frac{\delta\theta}{2} \doteq \frac{\delta\theta}{2} \text{ when the angle is small}\right)$$

In the limit $$P = Td\theta \quad \text{. . . . . (2)}$$

Then, combining equations (1) and (2),

$$dT = \mu T d\theta$$

Separating the variables, $$\int_{T_2}^{T_1}\frac{dT}{T} = \mu\int_0^{\theta} d\theta$$

Integrating, $$\log_e\frac{T_1}{T_2} = \mu\theta$$

or $$\frac{T_1}{T_2} = e^{\mu\theta}$$

Friction in a Footstep Bearing.

Example 28.—To find the moment of the friction force in a footstep bearing; the coefficient of friction being μ, $R =$ radius of journal and $W =$ total load.

(*a*) Assume that the pressure is uniform over the bottom surface, *i. e.*, $W = \pi R^2 p$, where p is the intensity of the pressure.

Take an annulus at radius r, and of thickness δr (Fig. 124).

Area of the annulus $= 2\pi r\delta r$
Pressure on the annulus $= 2\pi r\delta r p$
Friction force on the annulus $= 2\pi r\delta r\mu p$

and hence the moment of the friction force on the annulus

$$= 2\pi r\,\delta r\,\mu p \times r$$

and the total moment of the friction force $= \int_0^R \mu p \, 2\pi r^2 dr$

$$= 2\pi\mu p \frac{R^3}{3}$$

$$= \frac{2}{3}\pi\mu R^3 . \frac{W}{\pi R^2}$$

$$= \frac{2}{3} R \times \mu W$$

i. e., the moment is the same as it would be if the whole load were supposed concentrated at a distance of two-thirds of the radius from the centre.

(*b*) Assume that the intensity of pressure varies inversely as the velocity,

i. e., $p = K \times \frac{1}{v}$

the velocity at radius r $= v_r = 2\pi n r$

so that $p = K \times \frac{1}{2\pi n r} = \frac{m}{r}$ (say).

Then the pressure intensity on an annulus distant r from the centre

$$= p = \frac{m}{r}$$

and the total pressure on the annulus

$$= 2\pi r \delta r \times \frac{m}{r}$$

$$= 2\pi m \, \delta r$$

also the friction force $= \mu \times$ this pressure.

Hence the moment of the friction force on the annulus

$$= 2\pi\mu m \, r\delta r$$

and the total moment of the friction force

$$= \int_0^R 2\pi\mu m r dr$$

$$= \frac{2\pi\mu m R^2}{2}.$$

Now the total load W $= \int_0^R$ intensity $\times$ area

$$= \int_0^R p \times 2\pi r dr$$

$$= \int_0^R \frac{2\pi r dr \times m}{r}$$

$$= 2\pi m R.$$

Hence the moment of the friction force

$$= 2\pi m R \times \frac{\mu R}{2}$$

$$= \mu W \times \frac{R}{2}$$

i. e., the effective radius is now $\frac{1}{2}$ and not $\frac{2}{3}$, as in Case (*a*).

Example 29.—To find an expression for the moment of the friction force for a Schiele Pivot.

Assume that the pressure is the same all over the rubbing surface, that the wear is uniform and that the normal wear is proportional to the pressure p and to the speed v.

Referring to Fig. 125, $\delta n =$ normal wear $\propto pv$, *i. e.*, $2\pi nrp$,
and thus $\delta n \propto pr$ or $\delta n = Kpr$.

Let the tangent at the point P make the angle θ with the axis, then if $t =$ length of tangent, $t \sin\theta = r$.

Now $\delta h =$ vertical drop $= \dfrac{\delta n}{\sin\theta}$

or $\delta n = \delta h \sin\theta$

Also $\delta n = Kpr = Kpt \sin\theta$

whence $\delta h = Kpt$.

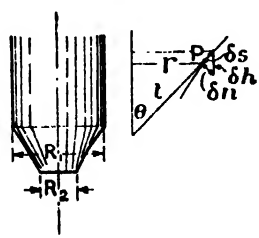


FIG. 125.

Now δh is constant, p and K are constant; hence t must be constant and the curve is that known as a tractrix (*i. e.*, the length of the tangent from the axis to any point on the curve is a constant).

To find the moment of the friction force:—

On a small element of surface, the friction force

$$= 2\pi r\, \delta s \times p \times \mu$$

and the moment of the friction force

$$= 2\pi r \delta s\, \mu p \times r.$$

Now $\delta r = \delta s \sin\theta$.

Hence the total moment of the friction force

$$= \int_{R_2}^{R_1} 2\pi\mu p r \frac{dr}{\sin\theta} t \sin\theta$$

$$= 2\pi\mu p t \times \tfrac{1}{2}(R_1^2 - R_2^2)$$

but $$p = \frac{W}{\pi(R_1^2 - R_2^2)}.$$

Hence the total moment of the friction force $= \mu W t$.

Examples on Hydraulics.

Example 30.—To find the time to empty a tank, of area A sq. ft., through an orifice of area a sq. ft., the coefficient of discharge being C_d.

If the height of the water above the orifice at any time is h, then the velocity of discharge $= v = \sqrt{2gh}$.

Hence the quantity per sec. $= C_d av$

and the quantity in time $\delta t = C_d av\delta t$.

This flow will result in a lowering of the level in the tank by an amount δh, so that the volume taken from the tank in time $\delta t = A\,\delta h$.

Hence $$A\delta h = C_d a\sqrt{2gh}\delta t.$$

Here we have a simple differential equation to solve, and separating the variables and integrating

$$\int_0^t dt = \int_{h_1}^{h_2} \frac{A\,dh}{C_d a\sqrt{2g}\,h^{\frac{1}{2}}} \qquad \text{where } h_2 = \text{initial height}, \quad h_1 = \text{final height}$$

$$\therefore \qquad t = \frac{A}{a} \times \frac{1}{C_d\sqrt{2g}} \times 2\{\sqrt{h_2} - \sqrt{h_1}\}$$

$$= \frac{2A}{C_d a\sqrt{2g}}\{h_2^{\frac{1}{2}} - h_1^{\frac{1}{2}}\}.$$

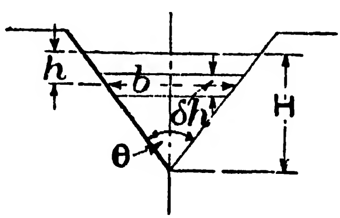


Fig. 126.—Triangular Notch.

If $h_1 = 0$, then the time to completely empty the tank

$$= \frac{2A\sqrt{h}}{C_d a\sqrt{2g}}.$$

Example 31.—To gauge the flow of water by measurements with a triangular notch.

Let the height at the notch be H, and consider a small element of width b, thickness δh, and height above the apex of the notch $(H-h)$.

From Fig. 126, $\frac{b}{2} = (H-h)\tan\frac{\theta}{2}$ where $\theta =$ the angle of the notch,

whence $$b = 2(H-h)\tan\frac{\theta}{2}.$$

Now the area of the element $= b\,\delta h$

and the velocity of water at that height $= \sqrt{2g \times \text{height}} = \sqrt{2gh}$.

Hence the actual quantity flowing $= C_d \times 2(H-h)\tan\frac{\theta}{2} \times \sqrt{2g}\,h^{\frac{1}{2}}\delta h$

and the total quantity flowing for the height H

$$= \int_0^H 2\sqrt{2g}C_d \tan\frac{\theta}{2}(H-h)h^{\frac{1}{2}}dh$$

$$= 2\sqrt{2g}C_d \tan\frac{\theta}{2}\int_0^H (Hh^{\frac{1}{2}} - h^{\frac{3}{2}})dh$$

$$= 2\sqrt{2g}\,C_d \tan\frac{\theta}{2}(\tfrac{2}{3}H^{\frac{5}{2}} - \tfrac{2}{5}H^{\frac{5}{2}})$$

$$= \frac{8}{15}\sqrt{2g}\,C_d \tan\frac{\theta}{2}\,H^{\frac{5}{2}}.$$

If $\theta = 90^\circ$ (a common case), $\tan\frac{\theta}{2} = 1$,

and then the discharge $= \frac{8}{15}\sqrt{2g}\,C_d H^{\frac{5}{2}} = 2{\cdot}66\,H^{\frac{5}{2}}$ if $C_d = {\cdot}62$.

Example 32.—To estimate the friction on a wheel disc revolving in a fluid.

Let the friction per sq. ft. $= fv^x$ and let the disc (of inside radius R_2 and outside radius R_1) revolve at n revs. per sec.

The velocity of an annulus at radius $r = 2\pi nr$

and thus the friction force per sq. ft. on this annulus

$$= (2\pi nr)^x f.$$

Hence the moment of the friction force on the annulus

$$= f(2\pi nr)^x \times 2\pi r\delta r \times r$$

$$= f \times (2\pi)^{x+1} n^x r^{x+2}\delta r$$

and the total moment of the friction force on one side of the disc $= M$

$$= \int_{R_2}^{R_1} f \times (2\pi)^{x+1} n^x r^{x+2}\,dr$$

$$= \frac{f \times (2\pi)^{x+1} \times n^x}{x+3}\{R_1^{x+3} - R_2^{x+3}\}.$$

The total moment (*i. e.*, on the two sides) $= 2M$

and H.P. lost in friction $= \dfrac{4\pi nM}{550}$.

If $x = 2$

$$M = \frac{f \times (2\pi)^3 \times n^2}{5}\{R_1^5 - R_2^5\}$$

$$= 49{\cdot}6fn^2\{R_1^5 - R_2^5\}.$$

Example 33.—**To establish a general rule for determining the depth of the Centre of Pressure of a section below the S.W.S.L.** (still water surface level).

Suppose the plate (representing a section) is placed as shown in Fig. 127. Consider a small element of area δa, distant x from OY, the vertical distance being h.

Let $\overline{X}$ = distance of the C. of G. from OY,

$\overline{\overline{X}}$ = distance of the C. of P. from OY,

and let $\overline{H}$ and $\overline{\overline{H}}$ be the corresponding vertical distances.

Let P = total pressure and A = total area,

k = swing radius about OY,

and ρ = weight of 1 cu. ft. of water.

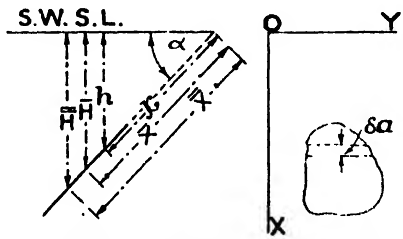


FIG. 127.—Centre of Pressure.

The whole pressure on the element

$$= \text{intensity of pressure} \times \text{area} = \rho h \times \delta a$$
$$= \rho x \sin a \delta a.$$

Whole pressure on surface $= \Sigma \rho x \sin a \delta a$ approx.

or $\int \rho x \sin a \, da$ actually,

i. e., $\mathrm{P} = \rho \sin a \times \int x \, da$

$= \rho \sin a \times$ 1st moment of area about OY

$= \rho \sin a \times \mathrm{A}\overline{\mathrm{X}}$,

but $\overline{\mathrm{X}} \sin a = \overline{\mathrm{H}}$.

$\therefore$ $\mathrm{P} = \rho \overline{\mathrm{H}} \mathrm{A}$.

To find the position of the C. of P., take moments about OY.

Then $\mathrm{P} \times \overline{\overline{\mathrm{X}}} = \Sigma$ moments of the pressures on the elements

$= \Sigma \rho \sin a \, x \delta a \times x$

$= \rho \sin a \Sigma x^2 \delta a$ approx.

$= \rho \sin a \int x^2 \, da$ actually

$= \rho \sin a \times$ 2nd moment about OY

$= \rho \sin a \times \mathrm{A}k^2$.

Now $\qquad P = \rho\bar{H}A$

so that $\qquad \rho\bar{H}A\bar{\bar{X}} = \rho \sin a A k^2,$

i. e., $\qquad \bar{H}\bar{\bar{X}} = \sin a \,.\, k^2$

but $\qquad \bar{H} = \bar{X} \sin a$

hence $\qquad \bar{X}\bar{\bar{X}} = k^2.$

Thus if $\bar{X}$ is known, $\bar{\bar{X}}$ can be calculated.

If the body is not symmetrical, then $\bar{\bar{Y}}$ (the distance of the C of P. from OX) must be found by taking moments about OX.

In a great number of cases $a = 90°$, so that $\sin a = 1$, and thus

$$\bar{H} \times \bar{\bar{H}} = k^2.$$

Example 34.—A triangular plate is placed with its base along the S.W.S.L., the plate being vertical. Find the depth of the centre of pressure below the surface.

For this section, I about S.W.S.L. $= \frac{1}{12} bh^3$

and thus $\qquad k^2 = \dfrac{\frac{1}{12} bh^3}{\frac{1}{2} bh} = \dfrac{h^2}{6}$

also $\qquad \bar{H} = \frac{1}{3} h.$

Hence $\qquad \bar{\bar{H}} = \dfrac{k^2}{\bar{H}} = \dfrac{h^2 \times 3}{6 \times h} = \underline{\dfrac{h}{2}}$

Example 35.—A circular plate has its upper edge along the S.W.S.L. Find the depth of its Centre of Pressure below the S.W.S.L.

For a circle, $\qquad I_{\text{diam.}} = \dfrac{\pi}{64} d^4$

and thus $\qquad k^2$ about diam. $= \dfrac{\frac{\pi}{64} d^4}{\frac{\pi}{4} d^2} = \dfrac{d^2}{16}$

and hence, by the parallel axis theorem,

$$k^2 \text{ about S.W.S.L.} = \frac{d^2}{16} + \left(\frac{d}{2}\right)^2 = \frac{5d^2}{16}$$

also $\qquad \bar{H} = \dfrac{d}{2}$

Hence $\qquad \bar{\bar{H}} = \dfrac{k^2}{\bar{H}} = \dfrac{\frac{5d^2}{16}}{\frac{d}{2}} = \underline{\dfrac{5d}{8}}$

Example 36.—**Forced vortex** (*i. e.*, **water in a tube rotated round a vertical axis). To find the form taken by the surface of the water.**

Let the rotation be at n R.P.S.

Consider an element at P (Fig. 128).

Then $\tan\theta =$ the slope of the curve taken $= \dfrac{dr}{dh}$

and also $$\tan\theta = \frac{\text{vertical force}}{\text{horizontal force}} = \frac{\text{weight of particle}}{\text{centrifugal force on particle}}$$

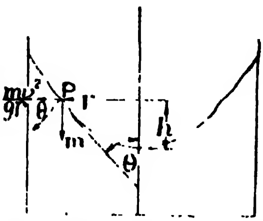


FIG. 128.—Forced Vortex.

$$= \frac{m \times gr}{mv^2}$$

$$= \frac{gr}{v^2}$$

$$= \frac{gr}{(2\pi nr)^2}$$

$$= \frac{g}{4\pi^2 n^2 r}$$

thus $$\frac{dr}{dh} = \frac{g}{4\pi^2 n^2 r}.$$

Separating the variables

$$4\pi^2 n^2 r\, dr = g dh.$$

Integrating $$\frac{4\pi^2 n^2 r^2}{2} = gh.$$

$\therefore$ $$h = \frac{2\pi^2 n^2}{g} \times r^2.$$

Now $\dfrac{2\pi^2 n^2}{g}$ is a constant, and thus

$h = \text{constant} \times r^2$, this being the equation of a parabola.

Hence the surface of the liquid will be that of a paraboloid of revolution.

An Example from Surveying.

Example 37.—Prove that a cubic parabola is a suitable "*transition*" curve.

In order that the full curvature of a railway curve may be approached gradually, a curve known as a transition curve is interposed between the straight and the curve. It must be so designed that the radius of curvature varies inversely as the distance from the starting point (on the straight, because there the radius is infinite).

As before proved, $$\frac{1}{R} = \frac{d^2y}{dx^2}$$

or more exactly $$\frac{\dfrac{d^2y}{dx^2}}{\left(1 + \left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$$

For the cubic parabola we may assume an equation

$$y = px^3,$$

where x is the distance from the straight, along a tangent, and y is the offset there (obviously p must be very small).

If $$y = px^3, \quad \frac{dy}{dx} = 3px^2, \quad \frac{d^2y}{dx^2} = 6px.$$

Hence $$\frac{1}{R} = \frac{6px}{(1+9p^2x^4)^{\frac{3}{2}}} = 6px(1+9p^2x^4)^{-\frac{3}{2}}.$$

$$\simeq 6px(1-13{\cdot}5p^3x^6)$$

as a first approximation, or $\frac{1}{R} \simeq 6px$ nearly (for p^3 is very small).

Hence $$\frac{1}{R} = 6px \quad \text{or} \quad R = \frac{1}{6p} \times \frac{1}{x} = \frac{K}{x}$$

or $$R \propto \frac{1}{x}.$$

Exercises 23.

1. A cylindrical tank is kept full of water by a supply. Show that the time required to discharge a quantity of water, equal to the capacity of the tank, through an orifice in the bottom equals half the time required to empty the tank when the supply is cut off.

A tank 10 ft. high and 6 ft. diam. is filled with water. Find the theoretical time of discharge through an 8″ diam. orifice in the bottom.

2. A tank empties through a long pipe discharging into the air. If the head lost in the pipe is written $h_l = \frac{Kv^2}{2g}$, show that K can be found from the expression,

$$t_1 - t_2 = 2\frac{A}{a}\sqrt{\frac{1+K}{2g}}\left(h_1^{\frac{1}{2}} - h_2^{\frac{1}{2}}\right)$$

where h_1 is the level of the water in the tank at the time t_1
and h_2 is the level of the water in the tank at the time t_2
measured from the centre of the discharge end of the pipe.

A = area of cross section of the tank.
a = area of cross section of the pipe.

An experiment with a tank 15·6 sq. ft. in cross section and a 4″ diam pipe gave the following results:

Time, t min.	Level in tank, h ft.
0	38·35
4	19·15

Find the value of K for the pipe.

3. Use the following table to obtain $\frac{dP}{dT}$ and thence find the volume of 1 lb. of steam at 160 lbs. absolute pressure per sq. in.

Absolute press. (lbs. per sq. in.)	159	160	161
Temperature (F°)	363·1	363·6	364·1

The latent heat of 1 lb. of steam at 160 lbs. per sq. in. pressure is 858·8 B.Th.U.

4. Find the "fixing moments" for a beam built in at its ends and 40 feet long, when it carries loads of 8 tons and 12 tons, acting 15 feet and 30 feet respectively from one end.

5. A tank of constant cross section has two circular orifices, each 2" diam., in one of its vertical sides, one of which is 20 ft. above the bottom of the tank and the other 8 ft.

Find the time required to lower the water from 30 ft. down to 15 ft. above the bottom of the tank.

Cross section of the tank = 12 sq. ft.
Coefficient of discharge = ·62.

6. A hemispherical tank 12 ft. in diam. is emptied through a hole 8" diam. at the bottom. Assuming that the coefficient of discharge is ·6, find the time required to lower the level of the water surface from 6 ft. to 4 ft.

7. A vertical shaft having a conical bearing is 9" in diam. and carries a load of $3\frac{1}{2}$ tons; the angle of the cone is 120° and the coefficient of friction is ·025. Find the horse power lost in friction when the shaft is making 140 revolutions per minute.

Assume that the intensity of pressure is uniform.

8. A circular plate, 5 ft. diam., is immersed in water, its greatest and least depths below the surface being 6 ft. and 3 ft. respectively; find

(*a*) the total pressure on one face of the plate,
(*b*) the position of the centre of pressure.

9. An annular plate is submerged in water in such a position that the minimum depth of immersion is 4 ft. and the maximum depth of immersion is 8 ft. If the external diam. of the plate is 8 ft. and the internal diam. 4 ft., determine the total pressure on one face of the plate and the position of the centre of pressure.

10. One pound of steam at 100 lbs. per sq. in. absol. (vol. = 4·45 cu. ft.) is admitted to a cylinder and is then expanded to a ratio of 5, according to the law $pv^{1.06} = C$; it is then exhausted at constant pressure.

Find the net work done on the piston.

11. Find the loss of head h in a length l of pipe the diameter of which varies uniformly, being given that

$$H = \frac{4fLv^2}{2gd}, \quad \text{and} \quad v = \frac{4Q}{\pi d^2}.$$

{Let diam. at distance x from entry end $= d_e + Kx$
where d_e = diam. at entry}.

12. Taking the friction of a brass surface in a fluid as ·22 lb. per sq. ft. for a velocity of 10 f.p.s. and as proportional to $v^{1\cdot 9}$, find the horse power lost in friction on two sides of a brass disc 30″ external and 15″ internal diam. running at 500 r.p.m.

13. A rectangular plate 2 ft. wide by 5 ft. deep is immersed in water at an inclination of 40° to the vertical. Find the depth of the centre of pressure, if the top of the plate is 6 ft. below the level of the water.

CHAPTER XI

HARMONIC ANALYSIS

Fourier's Theorem relates to periodic functions, of which many examples are found in both electrical and mechanical engineering theory and practice: it states that any periodic function can be expressed as the sum of a number of sine functions, of different amplitudes, phases and periods. Thus, however irregular the curve representing the function may be, so long as its ordinates repeat themselves after the same interval of time or space, it is possible to resolve it into a number of sine curves, the ordinates of which when added together give the ordinates of the primitive curve. This resolution of a curve into its component sine curves is known as *Harmonic Analysis*; and in view of its importance, the simpler and most direct methods employed for the analysis are here treated in great detail.

Expressed in mathematical symbols, Fourier's theorem reads

$$y = f(t) = A_0 + B \sin(\omega t + c_1) + C \sin(2\omega t + c_2) + \dots$$

or $$y = A_0 + A_1 \cos \omega t + A_2 \cos 2\omega t + A_3 \cos 3\omega t + \dots$$
$$+ B_1 \sin \omega t + B_2 \sin 2\omega t + B_3 \sin 3\omega t + \dots$$

the latter form being equivalent to the first, since

$$A_1 \cos \omega t + B_1 \sin \omega t \equiv B \sin(\omega t + c_1)$$

provided that B and c_1 are suitably chosen.

For the purposes of the analysis the expression may appear simpler if we write θ in place of ωt.

Thus

$$y = A_0 + A_1 \cos \theta + A_2 \cos 2\theta + A_3 \cos 3\theta + \dots$$
$$+ B_1 \sin \theta + B_2 \sin 2\theta + B_3 \sin 3\theta + \dots$$

Of the various methods given, two are here selected and explained, these being easy to understand and to apply.

Dealing with the two processes in turn, viz., (*a*) by calculation, and (*b*) by superposition, we commence with the study of method (*a*).

Method (*a*): Analysis by Calculation.—Before actually proceeding to detail the scheme of working, it is well to verify the following statements.

$\int_0^{2\pi} \cos\theta d\theta = 0$, this being self-evident, since the area under a cosine curve is zero, provided that the full period is considered.

$$\int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta = 0 \quad \ldots \quad (1)$$

for

$$\cos m\theta \cos n\theta = \tfrac{1}{2}\{\cos(m+n)\theta + \cos(m-n)\theta\}$$

and hence

$$\int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta = \tfrac{1}{2}\int_0^{2\pi} \cos(m+n)\theta \, d\theta + \tfrac{1}{2}\int_0^{2\pi} \cos(m-n)\theta \, d\theta$$
$$= 0 + 0$$

(for both are cosine curves over the full period or a multiple of the full period).

$$\int_0^{2\pi} \cos m\theta \sin n\theta \, d\theta = \tfrac{1}{2}\int_0^{2\pi} \sin(m+n)\theta \, d\theta - \tfrac{1}{2}\int_0^{2\pi} \sin(m-n)\theta \, d\theta$$
$$= 0 \quad \ldots \quad (2)$$

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \tfrac{1}{2}\int_0^{2\pi} \cos 2\theta \, d\theta + \tfrac{1}{2}\int_0^{2\pi} d\theta$$
$$= 0 + \tfrac{1}{2}(2\pi - 0) = \pi \quad \ldots \quad (3)$$

$$\int_0^{2\pi} \sin m\theta \sin n\theta \, d\theta = \tfrac{1}{2}\int_0^{2\pi} \cos(m-n)\theta \, d\theta - \tfrac{1}{2}\int_0^{2\pi} \cos(m+n)\theta \, d\theta$$
$$= \tfrac{1}{2}\{0 - 0\} = 0 \quad \ldots \quad (4)$$

and

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \tfrac{1}{2}\int_0^{2\pi} d\theta - \tfrac{1}{2}\int_0^{2\pi} \cos 2\theta \, d\theta$$
$$= \tfrac{1}{2}\{2\pi - 0\} - 0$$
$$= \pi \quad \ldots \quad (5)$$

To proceed with the analysis:—

We are told that

$$y = A_0 + A_1\cos\theta + A_2\cos 2\theta + A_3\cos 3\theta + \ldots$$
$$+ B_1\sin\theta + B_2\sin 2\theta + B_3\sin 3\theta + \ldots$$

and we wish to find the values of the coefficients A_0, $A_1 \ldots$ B_1, B_2, etc.

If we integrate throughout (with the limits 0 and 2π), every term on the right-hand side, except the first, will vanish,

i.e., $$\int_0^{2\pi} y\,d\theta = A_0\int_0^{2\pi} d\theta + 0 + 0 + \ldots$$

or $$\int_0^{2\pi} y\,d\theta = A_0 \times (2\pi - 0)$$

whence $$A_0 = \frac{1}{2\pi - 0}\int_0^{2\pi} y\,d\theta$$

$$= \text{the mean value of } y \quad \text{(Cf. p. 183)}$$

so that A_0 is found by averaging the ordinates; but in the majority of cases an inspection will show that A_0 is zero.

To find A_1:—multiply throughout by its coefficient, viz., $\cos\theta$, and integrate, then

$$\int_0^{2\pi} y\cos\theta\,d\theta = \int_0^{2\pi} A_0\cos\theta\,d\theta + \int_0^{2\pi} A_1\cos^2\theta\,d\theta + \int_0^{2\pi} A_2\cos\theta\cos 2\theta\,d\theta + \ldots$$
$$+\int_0^{2\pi} B_1\cos\theta\sin\theta\,d\theta + \int_0^{2\pi} B_2\cos\theta\sin 2\theta\,d\theta + \ldots$$

or $$\int_0^{2\pi} y\cos\theta\,d\theta = 0 + \pi A_1 + 0 + 0 \ldots$$
$$+ 0 + 0 \ldots \quad \text{[from (3), (2) and (1)]}$$

whence $$A_1 = \frac{2}{2\pi}\int_0^{2\pi} y\cos\theta\,d\theta$$

$$= \text{twice the mean value of } (y\cos\theta)$$

i. e., a certain number of values of y must be taken, each being multiplied by the cosine of the angle for which y is the ordinate, the average of these found, and the result multiplied by 2.

The values of A_2, A_3, etc., may be found in like manner by multiplying through by $\cos 2\theta$, $\cos 3\theta$, etc., in order, and performing the integration as above.

To find B_1:—multiply throughout by its coefficient, viz., $\sin\theta$, and integrate, then

$$\int_0^{2\pi} y\sin\theta\,d\theta = \int_0^{2\pi} A_0\sin\theta\,d\theta + \int_0^{2\pi} A_1\sin\theta\cos\theta\,d\theta + \int_0^{2\pi} A_2\sin\theta\cos 2\theta\,d\theta + \ldots$$
$$+\int_0^{2\pi} B_1\sin^2\theta\,d\theta + \int_0^{2\pi} B_2\sin\theta\sin 2\theta\,d\theta + \ldots$$

$$= 0 + \pi B_1 \quad \text{[from (2), (4) and (5)]}$$

$$\therefore\ B_1 = \frac{2}{2\pi}\int_0^{2\pi} y\sin\theta\,d\theta = 2 \times \text{mean value of } (y\sin\theta)$$

so that the values of B_1, B_2, B_3, etc., may be found.

To analyse a given curve the base is divided into a number of equal divisions, the number chosen being at least twice that of the highest harmonic occurring, and the ordinates tabulated.

Thus if it is expected that the highest harmonic will be the fourth, at least eight ordinates must be read.

Example 1.—From a graph the following values were read :—

0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
3·5	7·8	10·5	10	7·5		−1·5	−3·5	−4·5	−3·8	−2·3	0·5

Resolve this curve into its components, there being no higher harmonic than the third.

The ordinates given are very suitable and as the equation will be

$$y = A_0 + A_1 \cos\theta + A_2 \cos 2\theta + A_3 \cos 3\theta + B_1 \sin\theta + B_2 \sin 2\theta + B_3 \sin 3\theta$$

a table is made up thus :—

(a) θ	(b) y.	(c) $\sin\theta$	(d) $\cos\theta$	(e) $\sin 2\theta$	(f) $\cos 2\theta$	(g) $\sin 3\theta$	(h) $\cos 3\theta$
0	3·5	0	1	0	1	0	1
30°	7·8	·5	·866	·866	·5	1	0
60°	10·5	·866	·5	·866	−·5	0	−1
90°	10	1	0	0	−1	−1	0
120°	7·5	·866	−·5	−·866	−·5	0	1
150°	3	·5	−·866	−·866	·5	1	0
180°	−1·5	0	−1	0	1	0	−1
210°	−3·5	−·5	−·866	·866	·5	−1	0
240°	−4·5	−·866	−·5	·866	−·5	0	1
270°	−3·8	−1	0	0	−1	1	0
300°	−2·3	−·866	·5	−·866	−·5	0	−1
330°	·5	−·5	·866	−·866	·5	−1	0

Then A_0 = mean value of $y = \frac{1}{12}$ (sum of column (b)) $= 2{\cdot}27$

$A_1 = 2 \times$ mean value of $y \cos\theta$

$= \frac{1}{6} \times$ sum of products of columns (b) and (d)

$$= \frac{1}{6}[1(3{\cdot}5+1{\cdot}5)+{\cdot}866(7{\cdot}8-3+3{\cdot}5+{\cdot}5) + {\cdot}5(10{\cdot}5-7{\cdot}5+4{\cdot}5-2{\cdot}3)] = 2{\cdot}54$$

$A_2 = \frac{1}{6} \times$ sum of products of columns (b) and (f)

$$= \frac{1}{6}[1(3{\cdot}5-10-1{\cdot}5+3{\cdot}8)+{\cdot}5(7{\cdot}8-10{\cdot}5-7{\cdot}5 +3-3{\cdot}5+4{\cdot}5+2{\cdot}3+{\cdot}5)] = -{\cdot}99$$

$$A_3 = \frac{1}{6} \times \text{sum of products of columns } (b) \text{ and } (h)$$

$$= \frac{1}{6}[1(3{\cdot}5-10{\cdot}5+7{\cdot}5+1{\cdot}5-4{\cdot}5+2{\cdot}3)] = -{\cdot}03$$

In like manner

$$B_1 = \frac{1}{6}[1(10+3{\cdot}8)+{\cdot}866(10{\cdot}5+7{\cdot}5+4{\cdot}5+2{\cdot}3)$$
$$+{\cdot}5(7{\cdot}8+3+3{\cdot}5-{\cdot}5)] = 7{\cdot}03$$

$$B_2 = \frac{1}{6}[{\cdot}866(7{\cdot}8+10{\cdot}5-7{\cdot}5-3-3{\cdot}5-4{\cdot}5+2{\cdot}3-{\cdot}5)] = {\cdot}23$$

$$B_3 = \frac{1}{6}[1(7{\cdot}8-10+3+3{\cdot}5-3{\cdot}8-{\cdot}5)] = 0.$$

Hence

$$\underline{y = 2{\cdot}27+2{\cdot}54 \cos\theta - {\cdot}99 \cos 2\theta - {\cdot}03 \cos 3\theta + 7{\cdot}03 \sin\theta + {\cdot}23 \sin 2\theta.}$$

Method (*b*) : **Analysis by Superposition.**—This method is much used in alternating current work, for the problems of which it is specially suited. It is not difficult to employ, nor to understand, although the proof of the method is long and is in consequence not treated here.

In order to present the method in as clear a fashion as possible, the rules of procedure are here set out in place of a detailed explanation.

The method is as follows ; the case of a curve containing the third as the highest harmonic being treated, although the process can readily be extended if necessary :—

(1) Divide the curve into two equal parts and superpose the second part upon the first, using dividers and paying attention to the signs. If the resultant curve approximates to a sine curve there is no need to subdivide further. (This gives terms containing 2θ, 4θ 6θ, etc., but if this curve is a sine curve, probably only terms containing 2θ occur.)

Put in a base line for this new curve (by estimation) ; then the height of this from the original base line $= 2A_0$.

(2) Divide the original curve into three equal parts and superpose (first, the second on the first, and then to this result add the third).

(This gives the terms containing 3θ, 6θ, 9θ, etc.)

The height of the base line of the resulting curve from the original base line $= 3A_0$. (The two values of A_0 may be compared, and of course they should be alike ; but if not, take the average of these and draw a new base line distant A_0 from the original ; this line we shall speak of as the true base line.)

(3) Subtract corresponding ordinates of the 2θ curve (divided by 2) and the 3θ curve (divided by 3), paying attention to the

signs, from the ordinates of the original curve ; the resultant curve is approximately a sine curve symmetrical about the true base line.

To calculate the values of the constants, if

$$y = A_0 + A_1 \sin(\theta + c_1) + A_2 \sin(2\theta + c_2)$$

A_0 is already found.

Select two convenient values of θ and work from the ordinates of the θ curve to find A_1 and c_1; proceed similarly, using the 2θ curve to find A_2 and c_2.

Note that in alternating current work only terms of the order θ, 3θ, 5θ, etc., occur, so that the curve would need to be divided into 3, 5, etc., equal divisions and the parts superposed. There is thus no need to divide into 2, 4, etc., equal parts; also it is evident that the value of A_0 must be zero.

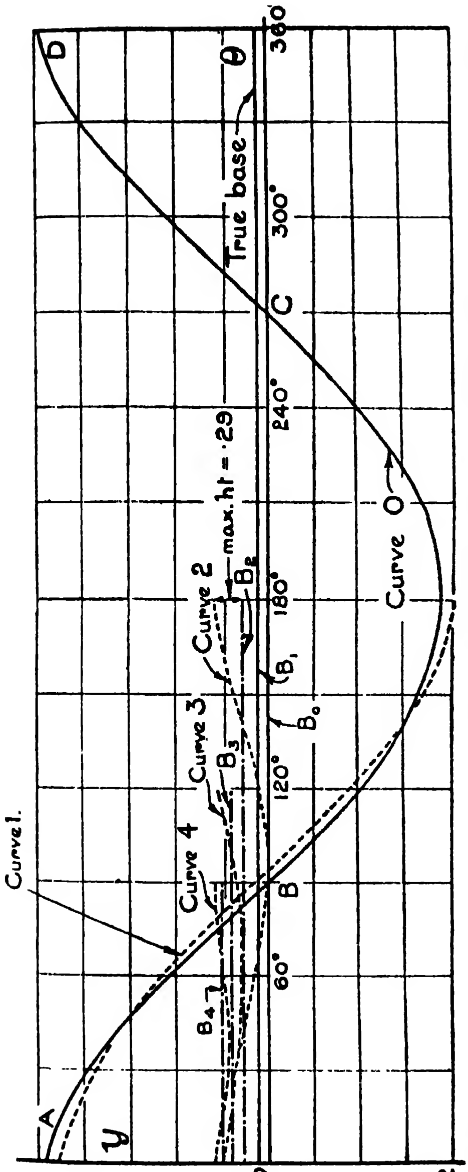


FIG. 129.

Example 2.—The curve ABCD, Fig. 129, gives by its ordinates the displacement of a valve actuated by a Gooch Link Motion.

It is required to find the constants in the equation

$$y = A_0 + A_1 \sin(\theta + c_1) + A_2 \sin(2\theta + c_2) \text{ etc.}$$

The original curve is divided into two equal parts, the second being placed over the first, with the result that Curve 2 is obtained.

The estimated base line for this is B_2; the height of B_2 above the original base line being ·29, *i. e.*,

the height of the true base line is $\frac{\cdot 29}{2}$ or $\cdot 145$ unit. This base line can now be put in, and is indicated as the true base line.

By division into 3 and 4 equal parts and superposition the curves 3 and 4 respectively are obtained.

B_3, the base line for 3, is at a height of $\cdot 43$; this figure divided by 3 gives $\cdot 143$, which agrees well with our former result.

Curve 2 really represents the second harmonic with double amplitude; therefore we subtract ordinates of Curve 2 (to half scale, *i.e.*, we use proportional compasses) from the corresponding ordinates of the original curve.

Similarly we subtract $\frac{1}{3}$ of the ordinates of Curve 3 from the original curve, and since those for Curve 4 are too small to be taken into account, the net result is Curve 1, which represents the fundamental, and is a sine curve symmetrical about the true base line.

To find the constants A_1 and c_1 in the equation

$$y_1 = A_1 \sin(\theta + c_1).$$

When $\theta = 0$, $y_1 = 2\cdot 175$ (measured from the true base line to Curve 1).

At $\theta = 90°$, $y_1 = 0$.

$\therefore$ $c_1 = 90°$ or $\frac{\pi}{2}$.

At $180°$ $y_1 = -2\cdot 135$,

i.e., $A_1 =$ the mean of $2\cdot 175$ and $2\cdot 135$, *i.e.*, $2\cdot 15$.

$$\therefore \quad y_1 = 2\cdot 15 \sin(\theta + 90°)$$
$$= 2\cdot 15 \cos\theta.$$

To find A_2 and c_2.

The amplitude of Curve 2 is $\frac{\cdot 29}{2}$, *i.e.*, $\cdot 145$.

$$\therefore \quad A_2 = \cdot 145,$$

and since the curve has its maximum ordinate when $\theta = 0$ we have again $c_2 = 90$, or the curve is a cosine curve.

Hence $y_2 = \cdot 145 \cos 2\theta$.

Beyond this first harmonic we need not proceed as the amplitudes of Curves 3 and 4 are exceedingly small.

Hence
$$y = y_1 + y_2$$
$$\underline{= 2\cdot 15 \cos\theta + \cdot 145 \cos 2\theta.}$$

This method of superposition is to be recommended in cases of A.C. work, as one can so readily tell by its aid which harmonics are present. If the actual constants in the equation are required it may be easier to proceed according to method (*a*).

Fourier Series.—It is sometimes necessary to determine a Fourier series to represent a given algebraic or trigonometric function over a stated range.

If $f(x)$ is the function we may write

$$f(x) = A_1 \sin x + A_2 \sin 2x + \ldots + \frac{B_0}{2} + B_1 \cos x + B_2 \cos 2x + \ldots$$

$$= \sum_{n=1}^{n=\infty} A_n \sin nx + \sum_{n=0}^{n=\infty} B_n \cos nx$$

where A_n = twice the mean value of $f(x) \sin nx$,
B_n = ,, ,, ,, ,, ,, $f(x) \cos nx$.

The work of finding the coefficients may be reduced by noting that if $f(x)$ is an "even" function, *i. e.*, if $f(-x) = f(x)$, the series will consist of cosine terms only and if $f(x)$ is an "odd" function, *i. e.*, if $f(-x) = -f(x)$, no cosine terms will appear.

Example 3.—Find a series to represent a function which has the value $\frac{\pi}{4}$ from $x = 0$ to $x = \pi$, and $-\frac{\pi}{4}$ from $x = \pi$ to $x = 2\pi$.

This is clearly an "odd" function so that there will be sine terms only.

$$A_n = \frac{2}{2\pi}\int_0^{2\pi} f(x) \sin nx\, dx = \frac{1}{\pi}\left[\int_0^{\pi} \frac{\pi}{4} \sin nx\, dx - \int_{\pi}^{2\pi} \frac{\pi}{4} \sin nx\, dx\right]$$

$$= \frac{1}{4}\left[-\frac{1}{n}(\cos nx)_0^{\pi} + \frac{1}{n}(\cos nx)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{4n}\Big[(\cos 0^\circ - \cos n\pi) + (\cos 2n\pi - \cos n\pi)\Big]$$

When n is even, $A_n = \frac{1}{4n}\Big[(1-1)+(1-1)\Big] = 0.$

When n is odd, $A_n = \frac{1}{4n}\Big[(1+1)+(1+1)\Big] = \frac{1}{n}.$

Hence $\underline{f(x) = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots}$

Example 4.—Find a series to represent $f(x)$ where $f(x) = \cos x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$ and 0 from $x = \frac{\pi}{2}$ to $x = \frac{3\pi}{2}$.

This is an even function and there will be no sine terms.

$$B_n = \frac{2}{2\pi}\int_{-\pi/2}^{\pi/2} \cos x \cos nx\, dx \text{ and for } n = 1,\ B_1 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos^2 x\, dx = \frac{1}{2}.$$

For other values of n

$$\begin{aligned} B_n &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\{\cos(n+1)x+\cos(n-1)x\}\,dx \\ &= \frac{1}{2\pi}\left[\frac{1}{n+1}\Big\{\sin(n+1)x\Big\}_{-\pi/2}^{\pi/2}+\frac{1}{n-1}\Big\{\sin(n-1)x\Big\}_{-\pi/2}^{\pi/2}\right] \\ &= \frac{1}{2\pi}\left[\frac{1}{n+1}\left(2\sin(n+1)\frac{\pi}{2}\right)+\frac{1}{n-1}\left(2\sin(n-1)\frac{\pi}{2}\right)\right]. \end{aligned}$$

If $n = 0, 4, 8 \ldots$

$$B_n = \frac{1}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}\right] = \frac{1}{\pi}\left(\frac{-2}{(n+1)(n-1)}\right).$$

If $n = 2, 6, 10 \ldots$

$$B_n = \frac{1}{\pi}\left[\frac{-1}{n+1}+\frac{1}{n-1}\right] = \frac{1}{\pi}\left(\frac{2}{(n+1)(n-1)}\right).$$

If $n = 3, 5, 7 \ldots$

$$B_n = 0.$$

Hence $$\underline{f(x) = \frac{1}{\pi}+\frac{1}{2}\cos x+\frac{2}{\pi}\left(\frac{\cos 2x}{1.3}-\frac{\cos 4x}{3.5}+\frac{\cos 6x}{5.7}\ldots\right)}$$

Example 5.—Find a series to represent e^x over the range $x = -2$ to $x = 2$.

Two results of integration by parts are quoted since they are used below :—

$$\int e^x \cos nx\,dx = \frac{e^x}{n^2+1}(\cos nx+n\sin nx)$$

$$\int e^x \sin nx\,dx = \frac{e^x}{n^2+1}(\sin nx-n\cos nx).$$

As the range of x is from -2 to 2 we may take the period of the fundamental as 4, so that

$$f(x) = A_1\sin\frac{\pi x}{2}+A_2\sin\pi x+\ldots+\frac{B_0}{2}+B_1\cos\frac{\pi x}{2}+B_2\cos\pi x+\ldots$$

$$\begin{aligned} B_n &= \frac{2}{4}\int_{-2}^{2}e^x\cos\frac{n\pi x}{2}\,dx = \frac{1}{2(n^2+1)}\left[e^x\left(\cos\frac{n\pi x}{2}+n\sin\frac{n\pi x}{2}\right)\right]_{-2}^{2} \\ &= \frac{1}{2(n^2+1)}\Big[e^2(\cos n\pi+n\sin n\pi)-e^{-2}(\cos n\pi-n\sin n\pi)\Big]. \end{aligned}$$

If n is even $\quad B_n = \frac{1}{2(n^2+1)}(e^2\,.\,1-e^{-2}\,.\,1) = \frac{\sinh 2}{n^2+1}.$

If n is odd $\quad B_n = \frac{1}{2(n^2+1)}(e^2\,.\,-1-e^{-2}\,.\,-1) = -\frac{\sinh 2}{n^2+1}.$

$$\begin{aligned} A_n &= \frac{2}{4}\int_{-2}^{2}e^x\sin\frac{n\pi x}{2}\,dx = \frac{1}{2(n^2+1)}\left[e^x\left(\sin\frac{n\pi x}{2}-n\cos\frac{n\pi x}{2}\right)\right]_{-2}^{2} \\ &= \frac{1}{2(n^2+1)}\Big[e^2(\sin n\pi-n\cos n\pi)-e^{-2}(-\sin n\pi-n\cos n\pi)\Big]. \end{aligned}$$

If n is even $A_n = \frac{1}{2(n^2+1)}\left[e^2(0-n)-e^{-2}(0-n)\right] = -\frac{n}{n^2+1}\sinh 2.$

If n is odd $A_n = \frac{1}{2(n^2+1)}\left[e^2(0+n)-e^{-2}(0+n)\right] = \frac{n}{n^2+1}\sinh 2.$

Hence

$$f(x) = \sinh 2\left(\frac{1}{2}+\frac{1}{2}\sin\frac{\pi x}{2}-\frac{2}{5}\sin\pi x+\frac{3}{10}\sin\frac{3\pi x}{2}\ldots\right.$$
$$\left.-\frac{1}{2}\cos\frac{\pi x}{2}+\frac{1}{5}\cos\pi x-\frac{1}{10}\cos\frac{3\pi x}{2}\ldots\right)$$

Exercises 24.—On Harmonic Analysis.

1. Show how to analyse approximately the displacement x of a point in a mechanism on the assumption that it may be represented by a limited series of sine and cosine terms, and obtain general expressions for the values of the coefficients in the series

$$x = \Sigma_1^n (A_n \cos n\theta + B_n \sin n\theta) + A_0$$

where $n = 3$ and θ is the angular displacement of an actuating crank which revolves uniformly. Apply your results to obtain the values of the coefficients for the values of x and θ given in the accompanying table, where the linear displacement of a point in a mechanism is given for the corresponding angular displacement of a uniformly revolving crank.

Angular displacement of crank in degrees.	0	60	90	120	180	240	300
x (in ins.)	1·11	·43	·83	1·65	3·67	4·15	2·93

2. A part of a machine has an oscillating motion. The displacements y at times t are as in the table.

t	·02	·04	·06	·08	·1	·12	·14	·16	·18	·2
y	·64	1·13	1·34	·95	0	−·92	−1·33	−1·16	−·66	0

Find the constants in the equation

$$y = A\sin(10\pi t + a_1) + B\sin(20\pi t + a_2).$$

3. Analyse the curve which results when the following values are plotted and ordinates are read at $x = 0°, 30°, 60° \ldots 330°$.

$x°$	0	45	90	135	180	225	270	315	360
y	0	21·5	31·25	11·25	0	9	30	26·5	0

4. The values of the primary E.M.F. of a transformer at different points in the cycle are as follows (θ being written in place of pt for reasons of simplicity).

θ	0	30	60	90	120	150	180	210	240	270	300	330	360
E	70	886	1293	1400	1307	814	−70	−886	−1293	−1400	−1307	−814	70

If θ and E are connected by the equation

$$E = A \sin \theta + B \sin 3\theta + C \cos \theta + D \cos 3\theta$$

find the values of the constants A, B, C and D.

CHAPTER XII

THE SOLUTION OF SPHERICAL TRIANGLES

THE curvature of the earth's surface is not an appreciable factor in the calculations following a small survey, and is therefore not regarded, but when the lengths of the boundaries of the survey are great, as in the case of a "major triangulation," the effect of the curvature must be allowed for, if precision is desired. It is therefore necessary to use *Spherical Trigonometry* in place of the more familiar *Plane Trigonometry*, and accordingly a very brief chapter is inserted here, dealing mainly with the solution of spherical triangles.

Definitions of Terms used.—The earth may be considered as a sphere of radius 20,890,172 feet, this being the mean radius.

A *great circle* on a sphere is a circle traced by the intersection of the sphere by a plane passing through its centre; if the plane does not pass through the centre of the sphere, its intersection with the sphere is called a *small circle*. Thus all meridians are great circles, whilst parallels of latitude, except for the equator parallel, are all small circles.

A straight line on the earth's surface is in reality a portion of a great circle; hence a parallel of latitude is not a straight line, or, in other words, a movement due East or West is not a movement along a straight line.

A triangle set out on the earth's surface with straight sides is what is termed a "spherical triangle," its sides being arcs of great circles The lengths of these sides might be measured according to the usual rules, viz., in miles, furlongs, etc., but it is more usual to measure them by the sizes of the angles subtended by them at the centre of the sphere. In this connection it is convenient to remember that an arc of one nautical mile (6076 feet) subtends an angle of $1'$ at the centre of the earth; hence a length of 80 nautical miles would be spoken of as a side of $80'$, *i. e.*, $1° 20'$.

In Fig. 130 is shown the difference between great and small circles ; and AB, BC and CA being portions of great circles form a

spherical triangle (shown cross hatched). The length BA would be expressed by the magnitude of the angle BOA.

A spherical triangle ABC is shown in Fig. 131, O being the centre of the sphere. The arc AB is proportional to the angle AOB, and therefore, instead of speaking of AB as a length, it is quite legitimate to represent it by $\angle$ AOB.

c would thus stand for $\angle$ AOB, *b* for $\angle$ COA, and *a* for $\angle$ COB. As regards the angles of the triangle, the angle between CA and AB is that between the planes AOC and AOB and is, therefore, the angle between the tangents AD and AE. Spherical triangles should be regarded as the most general form of plane triangles; for if the radius of the sphere becomes infinite the spherical triangle becomes a plane triangle.

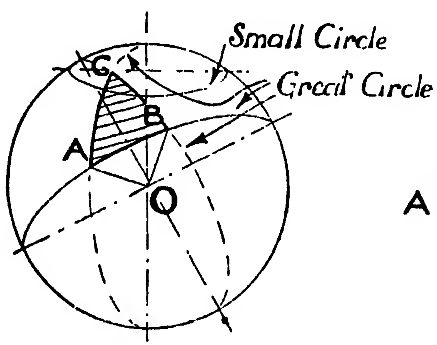


FIG. 130.

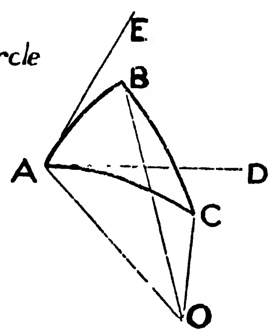


FIG. 131.

Spherical Triangles

Many rules with which we are familiar in connection with plane triangles hold also for spherical triangles, as, for example, "Any two sides of a triangle are greater than the third," or, again, "If two triangles have two sides and the included angle of the one respectively equal to two sides and the included angle of the other, the triangles are equal in all respects"; "The greater side of every triangle is subtended by the greater angle."

There is one important difference between the rule for a plane triangle and a corresponding rule for a spherical triangle: viz., whilst the three angles of a plane triangle add up to 180° the sum of those in a spherical triangle always exceeds 180°, the sum in fact lying between 180° and 540°; and the difference between the sum of the three angles and 180° is known as the "*spherical excess*."

The magnitude of this can be found from the rule

$$\text{spherical excess} = \frac{360^\circ \times \text{area of triangle}}{2\pi r^2}$$

($2\pi r^2$ being the area of the surface of the hemisphere).

This spherical excess is a small quantity for the cases likely to be considered in connection with surveys.

E. g., consider the case of an equilateral triangle of side 68 miles.

$$r = 20{,}900{,}000 \text{ ft. approx.} = 3960 \text{ miles.}$$

The area of the triangle is about 2000 sq. miles.

Then the spherical excess

$$= \frac{360 \times 60 \times 2000}{2\pi \times 3960 \times 3960} \text{ minutes} = \underline{\cdot 437 \text{ minute.}}$$

A good approximation for the spherical excess of a triangle on the earth's surface is:

$$\text{spherical excess (seconds)} = \frac{\text{area of spherical triangle in sq. miles}}{78}.$$

Solution of Spherical Triangles.—The most widely used rule in connection with the solution of plane triangles is the "*sine*" rule which states that the sides are proportional to the sines of the angles opposite. In the case of spherical triangles this becomes modified and reads—"**The sines of the angles are proportional to the sines of the sides opposite.**"

Therefore, adopting the notation of Fig. 131,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \quad \ldots\ldots \quad (1)$$

it being remembered that $\sin a$ is really $\sin \angle BOC$, etc.

Other rules are

$$\sin \frac{A}{2} = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}} \quad \ldots\ldots \quad (2)$$

$$\cos \frac{A}{2} = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}} \quad \ldots\ldots \quad (3)$$

$$\tan \frac{A}{2} = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}} \quad \ldots\ldots \quad (4)$$

and corresponding forms for $\frac{B}{2}$ and $\frac{C}{2}$, obtained by writing the letters one on in the proper sequence, $a\ b\ c\ a$.

s in these formulæ $= \frac{a+b+c}{2}$ and is, therefore, an angle

(in plane trigonometry, $s = \frac{a+b+c}{2}$, but is a length).

It is of interest to compare these with the corresponding rules in connection with plane triangles, which are

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

It will be seen that, as in the previous case, sides occurring in the formulæ of plane trigonometry are replaced by their sines in the corresponding formulæ of spherical trigonometry.

Other rules are :—

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\cot A \sin B = \cot a \sin c - \cos B \cos c \quad . \quad . \quad . \quad (7)$$

$$\tan\frac{A+B}{2} = \frac{\cos\frac{(a-b)}{2}}{\cos\left(\frac{a+b}{2}\right)}\cot\frac{C}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\tan\frac{A-B}{2} = \frac{\sin\frac{(a-b)}{2}}{\sin\left(\frac{a+b}{2}\right)}\cot\frac{C}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Solution of Right Angled Spherical Triangles.—In the case of a right angled spherical triangle these rules can be put into somewhat simpler forms.

Assume that the triangle is right angled at C.

$$C = 90^\circ, \quad \therefore \quad \cos C = 0, \quad \text{and} \quad \sin C = 1.$$

From (6) $$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$$

but $$\cos C = 0,$$

$$\therefore \quad \cos c - \cos a \cos b = 0,$$

i.e., $$\cos c = \cos a \cos b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Also $$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad \text{from (6)}$$

$$= \frac{\dfrac{\cos c}{\cos b} - \cos b \cos c}{\sin b \sin c} \quad \text{from (10)}$$

$$= \frac{\cos c}{\sin c} \times \frac{1}{\sin b \cos b} - \frac{\cos b}{\sin b} \times \frac{\cos c}{\sin c}$$

$$= \frac{1}{\tan c}\left\{\frac{1}{\sin b \cos b} - \frac{\cos b}{\sin b}\right\}$$

$$= \cot c\left\{\frac{1-\cos^2 b}{\sin b \cos b}\right\} = \frac{\cot c \times \sin^2 b}{\sin b \cos b}$$

$$= \cot c \times \frac{\sin b}{\cos b}$$

$$= \cot c \times \tan b$$

or $$\tan b \times \tan (90-c)$$

i. e., also
$$\left.\begin{aligned} \cos A &= \tan b \tan (90-c) \\ \cos B &= \tan a \tan (90-c) \end{aligned}\right\} \quad \dots\dots \quad (11).$$

Again $$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a - \dfrac{\cos^2 c}{\cos a}}{\sin b \sin c} \quad \text{from (10)}$$

$$= \frac{\cos^2 a - \cos^2 c}{\cos a \sin b \sin c} \quad \text{or} \quad \frac{\sin^2 c - \sin^2 a}{\cos a \sin b \sin c} \quad \dots \quad (12)$$

And from (1) $$\sin A = \frac{\sin a}{\sin c} \quad \dots\dots\dots \quad (13)$$

[In plane trigonometry $\sin A = \dfrac{a}{c}$].

Napier's Rules of Circular Parts.—The equations (10), (11) and (13) and their modifications may be easily remembered by *Napier's two rules of circular parts*, which may almost be regarded as a mnemonic.

For the application of these two rules the five parts of the spherical triangle, other than the right angle at C, are regarded as a, b, $(90-A)$, $(90-c)$, and $(90-B)$ respectively, the complements of A, c and B being taken instead of the values A, c and B in order that the two rules may embrace all the cases.

These five parts are written in the five sectors of a circle in the order in which they occur in a triangle: thus in Fig. 132, commencing from the side a and making the circuit of the triangle in the direction indicated, the parts in turn are a b A (for which we write $90-A$), c (for which is written $90-c$) and B (for which is written $90-B$). These parts are set out as shown in Fig. 133.

Then Napier's rules state :—

Sine of the middle part = product of tangents of adjacent parts.

Sine of the middle part = product of cosines of opposite parts.

The terms *middle*, *adjacent* and *opposite* have reference to the mutual position of the parts in Fig. 133. Thus if b is selected as the middle part, the adjacent parts are those in immediate contact

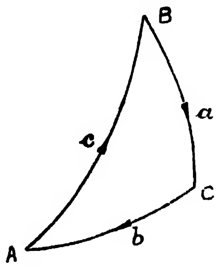


FIG. 132.

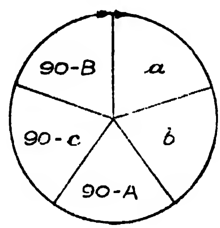


FIG. 133.

with b, viz., a and $(90-A)$, whilst $(90-c)$ and $(90-B)$ are the opposite parts.

Hence $\sin b = \tan a \times \tan(90-A) = \tan a \cot A$

$$\sin b = \cos(90-B) \times \cos(90-c) = \sin B \sin c$$

or $\sin B = \dfrac{\sin b}{\sin c}$. (Cf. equation (13), p. 359.)

Again if $(90-B)$ is selected as the middle part, the adjacent parts are $(90-c)$ and a, and the opposite parts are $(90-A)$ and b.

Hence $\sin(90-B) = \tan(90-c) \tan a$

or $\cos B = \tan(90-c) \tan a$ (cf. equation (11), p. 359),

and $\sin(90-B) = \cos(90-A) \cos b$

or $\cos B = \sin A \cos b$.

These rules, being composed of products and quotients only, lend themselves well to logarithmic computation.

The Ambiguous Case in the Solution of Spherical Triangles.—In the solution of a plane triangle, if two sides and the angle opposite the shorter of these is given, there is the possibility

of two solutions of the problem; the best test for which, as pointed out in Chap. VI, Part I, being the drawing to scale.

A similar difficulty occurs in the solution of spherical triangles, when two sides and the angle opposite one of them is given.

E. g., let a b and B be the given parts.

Then from equation (1), p. 357.

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b}$$

or
$$\sin A = \frac{\sin a \sin B}{\sin b}$$

Now $\sin A = \sin(180 - A)$ and thus the right-hand side of this last equation may be the value of either a particular angle or its supplement.

Without going into the proof it may be stated that there will be one solution only if the side opposite the given angle has a value between the other given side and its supplement. Thus in the case in which a b and B are given, there will be one solution only if b lies between a and $(180 - a)$.

If b is not between a and $(180 - a)$, then the test must be applied that the greater angle must be opposite the greater side: thus for the case of a b and B given, if $a > b$ then A must be $> B$. The possible cases may be best illustrated by numerical examples, a b and B being regarded as the given parts throughout.

(*a*) Given $a = 144°\,40'$, $b = 87°\,37'$, $B = 11°\,9'$ to find A.

Using equation (1) of p. 357

$$\sin A = \frac{\sin a \sin B}{\sin b} = \frac{\sin 144°\,40' \times \sin 11°\,9'}{\sin 87°\,37'}$$

and
$$\begin{aligned}\log \sin A &= \log \sin 144°\,40' + \log \sin 11°\,9' - \log \sin 87°\,37' \\ &= \bar{1} \cdot 7622 + \bar{1} \cdot 2864 - \bar{1} \cdot 9996 \\ &= \bar{1} \cdot 0490\end{aligned}$$

so that it is possible that A = either $6°\,26'$ or $173°\,34'$.

Now $a > b$ and therefore A must be $> B$, and this condition is only satisfied if $A = 173°\,34'$, since $6°\,26'$ is not $> 11°\,9'$.

It will be noted that the case chosen is that in which b, viz., $87°\,37'$, lies between a, *i. e.*, $144°\,40'$, and $(180 - a)$, *i. e.*, $35°\,20'$, and therefore only one solution is expected.

(*b*) $b = 44°\,35'$, $a = 55°\,10'$ and $B = 38°\,46'$.

Here b does not lie between a and $(180 - a)$, so that two solutions are possible.

As before

$$\log \sin A = \log \sin a + \log \sin B - \log \sin b$$
$$= \log \sin 55^\circ 10' + \log \sin 38^\circ 46' - \log \sin 44^\circ 35'$$
$$= \bar{1}\cdot 9142 + \bar{1}\cdot 7966 - \bar{1}\cdot 8463 = \bar{1}\cdot 8645$$
$$= \log \sin 47^\circ 3'$$

so that possible values of A are $47^\circ 3'$ and $132^\circ 57'$ and we must test each of these values.

Now $a > b$, and hence A must be $> B$; but $47^\circ 3'$ and $132^\circ 57'$ are both $> 38^\circ 46'$, so that we have two triangles satisfying the conditions, and for complete solution the two values of A, C and c must be determined.

Example 1.—In a spherical triangle ABC, having given $a = 30^\circ$, $b = 40^\circ$, $C = 70^\circ$, find A and B.

Given also that

$$\begin{cases} L\sin\ 5^\circ = 8{\cdot}9402960 \\ L\sin 35^\circ = 9{\cdot}7585913 \\ L\cos\ 5^\circ = 9{\cdot}9983442 \\ L\cos 35^\circ = 9{\cdot}9133645 \end{cases} \qquad \begin{aligned} &L\tan 12^\circ 14' 38'' = 9{\cdot}3364779 \\ &L\tan 60^\circ 4' 3'' = 10{\cdot}2397529 \end{aligned}$$

In this case two sides and the included angle are given; we therefore use equations (8) and (9).

$$\tan\frac{A+B}{2} = \frac{\cos\left(\frac{a-b}{2}\right)}{\cos\left(\frac{a+b}{2}\right)}\cot\frac{C}{2} \quad . \quad . \quad \text{from (8)}$$

$$= \frac{\cos\ 5^\circ}{\cos 35^\circ}\cot 35^\circ \qquad \left\{\begin{aligned}&\text{for } \cos -5^\circ \\ &\quad = \cos +5^\circ\end{aligned}\right\}$$

$$= \frac{\cos\ 5^\circ}{\cos 35^\circ} \times \frac{\cos 35^\circ}{\sin 35^\circ} = \frac{\cos\ 5^\circ}{\sin 35^\circ}$$

Taking logs of both sides

$$L\tan\frac{A+B}{2} = L\cos 5^\circ - L\sin 35^\circ + 10$$

(or, alternatively, $\log\tan\frac{A+B}{2} = \log\cos 5^\circ - \log\sin 35^\circ$)

$$\therefore \qquad L\tan\frac{A+B}{2} = \begin{array}{r} 19{\cdot}9983442 \\ \underline{9{\cdot}7585913} \\ 10{\cdot}2397529 \end{array}$$

$$= L\tan 60^\circ 4' 3''$$

$$\therefore \qquad \frac{A+B}{2} = 60^\circ 4' 3''$$

$$A+B = 120^\circ 8' 6'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

From equation (9)

$$\tan\frac{A-B}{2}=\frac{\sin\left(\frac{a-b}{2}\right)}{\sin\left(\frac{a+b}{2}\right)}\cot\frac{C}{2}$$

$\therefore$ $$\tan\frac{A-B}{2}=-\frac{\sin 5^\circ}{\sin 35^\circ}\,\frac{\cos 35^\circ}{\sin 35^\circ}$$

or $$\tan\frac{B-A}{2}=\frac{\sin 5^\circ\times\cos 35^\circ}{\sin^2 35^\circ}$$

taking logs throughout.

$$L\tan\frac{B-A}{2}=L\sin 5^\circ+L\cos 35^\circ-2L\sin 35^\circ+10$$

$$\begin{array}{r} 18{\cdot}9402960 \\ \underline{9{\cdot}91336 45} \\ =28{\cdot}8536 6605 \\ \underline{19{\cdot}5171826} \\ 9{\cdot}3364779 \\ =L\tan 12^\circ\,14'\,38'' \end{array}$$

$\therefore$ $B-A=24^\circ\,29'\,16''$ . . . . (*b*)

By adding (*a*) and (*b*) $2B=144^\circ\,37'\,22''$

$\therefore$ $B=\underline{72^\circ\,18'\,41''}$

and $A=120^\circ\,8'\,6''-72^\circ\,18'\,41''=\underline{47^\circ\,49'\,25''}$.

[Note that $A+B+C=47^\circ\,49'\,25''+72^\circ\,18'\,41''+70^\circ$

$=190^\circ\,8'\,6''$

so that the spherical excess $=10^\circ\,8'\,6''$.]

Example 2.—Solve the spherical triangle ABC, having given

$$c=91^\circ\,18',\quad a=72^\circ\,27',\quad\text{and}\quad C=90^\circ.$$

In this case the triangle is right angled, and therefore rules (10) to (13) may be used.

To find A :—

From equation (13), p. 359, $\sin A=\dfrac{\sin a}{\sin c}$.

$\therefore$ $L\sin A=L\sin a-L\sin c+10$

$=L\sin 72^\circ\,27'-L\sin 91^\circ\,18'+10$

$$\begin{array}{r} 19{\cdot}97930 \\ =\underline{9{\cdot}99989} \\ 9{\cdot}97941 \end{array}$$

$=L\sin 72^\circ\,29'\,45''$.

$\therefore$ $A=72^\circ\,29'\,45''$.

To find b :—

From equation (10), p. 358,

$$\cos c = \cos a \cos b$$

whence $$\cos b = \frac{\cos c}{\cos a}$$

$\therefore$ $$\cos b = \frac{\cos 91^\circ 18'}{\cos 72^\circ 27'} = \frac{-\cos 88^\circ 42'}{\cos 72^\circ 27'}$$

or $$\cos(180-b) = -\cos b = \frac{\cos 88^\circ 42'}{\cos 72^\circ 27'}.$$

Hence we shall work to find the supplement of b.

Taking logs *

$$\log\cos(180-b) = \log\cos 88^\circ 42' - \log\cos 72^\circ 27'$$

$$\begin{array}{l} \quad \bar{2}\cdot 35578 \\ = \underline{\bar{1}\cdot 47934} \\ \quad \bar{2}\cdot 87644 \end{array}$$

$$= \log\cos 85^\circ 41' 7''.$$

$\therefore$ $$180 - b = 85^\circ 41' 7''$$

$\therefore$ $$b = 94^\circ 18' 53''.$$

---

* It is rather easier to work in terms of the logs in preference to the logarithmic ratios. One must remember, however, that the L sine A = log sin A + 10, so that if a L sin A reading is 9·97941, then the reading for log sin A would be $\bar{1}\cdot 97941$. If the logarithmic ratios are used the addition of the 10 must not be overlooked.

To find B :—

From equation (11), p. 359,

$$\cos B = \tan a \tan(90^\circ - c)$$

i. e., $$\cos(180^\circ - B) = \tan a \tan(c - 90^\circ) = \tan 72^\circ 27' \times \tan 1^\circ 18'.$$

$\therefore$ $$\log\cos(180-B) = \log\tan 72^\circ 27' + \log\tan 1^\circ 18'$$

$$\begin{array}{l} \quad \cdot 49996 \\ = \underline{\bar{2}\cdot 35590} \\ \quad \bar{2}\cdot 85586 \end{array}$$

$$= \log\cos 85^\circ 53' 6''.$$

$\therefore$ $$180 - B = 85^\circ 53' 6''$$

$\therefore$ $$B = 94^\circ 6' 54''.$$

Hence, grouping our results,

$$\begin{array}{ll} a = 72^\circ 27' & A = 72^\circ 29' 45'' \\ b = 94^\circ 18' 53'' & B = 94^\circ 6' 54'' \\ \underline{c = 91^\circ 18'} & \underline{C = 90^\circ} \end{array}\Bigg\}$$

Example 3.—At a point A, in latitude 50° N., a straight line is ranged out which runs due E. at A. This straight line is prolonged for

60 nautical miles to B. Find the latitude of B, and if it be desired to travel due N. from B so as to meet the 50° parallel again at C, find the angle ABC at which we must set out and also the distance BC.

In Fig. 134 let A be the point on latitude 50° N. and ABD be a great circle passing through A: thus AB is a straight line running due E. from A. Let NB be the meridian through B, and NA that through A.

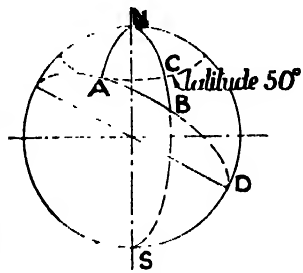


FIG. 134.

The sides NA, AB and BN are straight lines, because they are parts of great circles and therefore they together form a spherical triangle.

In this triangle we know the side AB (its value being 60′, for 1 nautical mile subtends an angle of 1′ at the centre); the angle at A (90°); and the side NA (90°—latitude, *i.e.*, 40°).

Thus two sides and the included angle are given and we require to solve the triangle; hence we use rule (10), p. 358,

from which $\cos NB = \cos NA \cos AB$

$$= \cos 40° \cos 60'$$

or $\log \cos NB = \log \cos 40° + \log \cos 1°$

$$= \bar{1}{\cdot}88425 + \bar{1}{\cdot}99993$$

$$= \bar{1}{\cdot}88418$$

i. e., $NB = 40° 0' 38''$

or the latitude of B is

$$90° - 40° 0' 38'' = \underline{40° 59' 22''}.$$

Now C is at the same latitude as A, so that BC is 38″, corresponding to $\frac{38}{60}$ nautical miles; *i. e.*, BC = ·633 nautical mile.

To find the angle ABC, we use rule (13), p. 359.

$$\sin \angle ABC = \frac{\sin NA}{\sin NB} = \frac{\sin 40°}{\sin 40° 0' 38''}$$

$\therefore$ $\log \sin \angle ABC = \log \sin 40° - \log \sin 40° 0' 38''$

$$= \bar{1}{\cdot}80807 - \bar{1}{\cdot}80817$$

$$= \bar{1}{\cdot}99990$$

whence $\angle ABC = \underline{88° 45'}.$

For the surveyor, spherical trigonometry has an important application in questions relating to spherical astronomy. Thus in the determination of the latitude of a place by observation to a star, the calculations necessary involve the solution of a spherical triangle. This triangle is indicated in Fig. 136, the sides AB, BC

and CA representing the co-latitude of the place, *i. e.*, (90°—latitude), (90°—declination) and (90°—altitude) respectively; whilst the angles A, B and C measure respectively the azimuth, the hour angle and the parallactic angle.

The terms just mentioned are defined as follows:—

Fig. 135 represents a section of the celestial sphere at the meridian through the point of observation O. RDT is the celestial equator, CEX is the horizon, Z is the *zenith* of the point of observation, *i. e.*, the point on the celestial sphere directly above O, and S marks the position of the heavenly body to which observa-

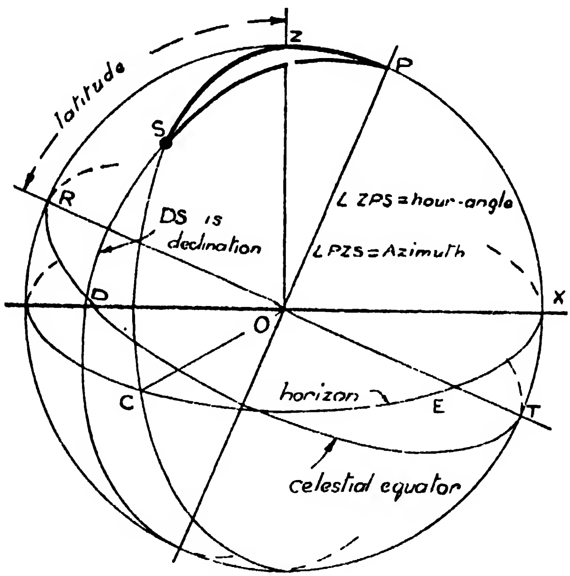


FIG. 135.—Determination of Latitude.

tions are made. Also PSD, ZSC, RDT and CEX are portions of great circles.

The *altitude* of a heavenly body is the arc of a great circle passing through the zenith of the point of observation and the heavenly body; the arc being that intercepted between the body and the horizon. We may thus compare the altitude in astronomy with the angle of elevation in surveying. Referring to Fig. 138, ZSC is the great circle passing through Z and S, and SC is the altitude. ZS, which is the complement of SC, is called the *zenith distance*.

The *azimuth* of a heavenly body is the angle between the meridian plane through the point of observation and the vertical plane passing through the body. It can be compared with the "bearing" of plane trigonometry. In Fig. 135, the angle PZS is the azimuth of S.

The *hour angle* of a heavenly body is the angle at the pole, between the meridian plane through the point of observation and the great circle through the pole and the body.

Thus, in the figure, P is the pole, and PSD is the great circle passing through P and S; this being known as the "declination circle." Then $\angle$ ZPS = the hour angle of S, and it is usually expressed in terms of time rather than in degrees.

The *declination* of a heavenly body is the arc of the declination

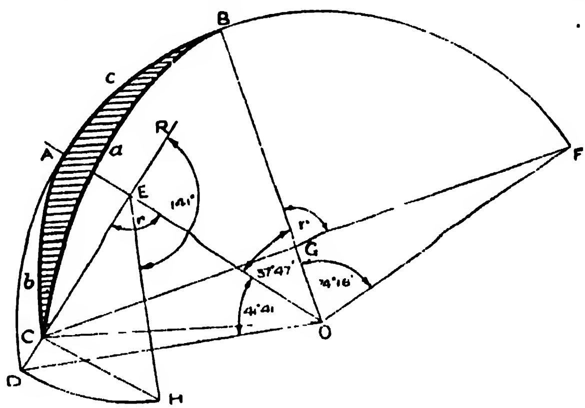


FIG. 136.

circle intercepted between the celestial equator and the heavenly body: thus DS is the declination of S.

The method of calculation can be best explained by working through a numerical example; and in order to ensure a clear conception of the problem, it is treated both graphically and analytically.

Example 4.—At a certain time at a place in latitude 52° 13′ N. the altitude of the Sun was found to be 48° 19′ and its declination was 15° 44′ N. Determine the azimuth.

As explained before, a spherical triangle can be constructed with sides as follows: a = 90—declination = 74° 16′, b = 90—altitude = 41° 41′, and c = co-latitude = 37° 47′. Then the angle A is the azimuth (Fig. 139).

Graphic construction.—With any convenient radius OD describe an arc of a circle DABF. Draw OA, OB and OF, making the angle DOA $= b = 41° 41'$, $\angle$ AOB $= c = 37° 47'$, and $\angle$ BOF $= a = 74° 16'$. Draw DCE at right angles to OA, and FGC at right angles to OB, intersecting at the point C. Note that C lies outside the triangle AOB. With centre E and radius ED construct the arc of a circle DH: draw CH perpendicular to DE to meet this arc at H and join EH. Then the angle REH is the value of the required angle A, and is found to be in the neighbourhood of 141°. [If C had fallen the other side of OA, the angle CEH would have been measured.]

The actual spherical triangle ABC is formed by the circular arc BA and the elliptical arcs AC and BC.

Proof of the construction—The side b is such that it subtends an angle of $41° 41'$ at the centre of the sphere. Thus DA measures the actual length of b, but does not represent it in its true position. In like manner BF gives the length of a, but again does not give its position on the sphere.

Let the circular sector OAD be rotated about OA as axis, and the sector OBF about OB as axis, and let the rotation of both be continued until they have a common radius OC, *i. e.*, OC is the intersection of the two revolving planes. Then evidently C is the third angular point of the spherical triangle ABC, since the given conditions concerning the lengths of the sides are satisfied by its position.

We observe that in this case the rotation of OBF has to be continued beyond OA, from which fact we gather that the angle at A must be obtuse. The line OA is in the plane of the paper, and taking a section along DE and turning this down to the plane of the paper, we observe that the actual height of C above the paper is CH. Thus EH is a line on the plane OAC, also ER is a line in the plane AOB, both lines being perpendicular to the line of intersection, and the angle REH therefore measures the inclination of the plane AOC to the plane AOB, this angle being by definition the angle A of the spherical triangle ABC.

By calculation.—Here we have the three sides given, and we wish to find an angle which may be done by use of equation (4), p. 357, viz.,

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}.$$

Now $s = \dfrac{a+b+c}{2} = \dfrac{74° 16' + 41° 41' + 37° 47'}{2} = \dfrac{153° 44'}{2} = 76° 52'$,

so that
$$s-a = 76° 52' - 74° 16' = 2° 36'$$
$$s-b = 76° 52' - 41° 41' = 35° 11'$$
$$s-c = 76° 52' - 37° 47' = 39° 5'.$$

Hence
$$\tan\frac{A}{2} = \sqrt{\frac{\sin 35° 11' \times \sin 39° 5'}{\sin 76° 52' \times \sin 2° 36'}}$$

and $\log\tan\dfrac{A}{2} = \dfrac{1}{2}\left[\begin{array}{r}(\log\sin 35^\circ 11' + \log\sin 39^\circ 5') \\ -(\log\sin 76^\circ 52' + \log\sin 2^\circ 36')\end{array}\right]$

$$= \frac{1}{2}\left[\begin{pmatrix}\bar{1}\cdot76057 \\ \underline{\bar{1}\cdot79965} \\ \bar{1}\cdot56022\end{pmatrix} - \begin{pmatrix}\bar{1}\cdot98849 \\ \bar{2}\cdot65670 \\ \bar{2}\cdot64519\end{pmatrix}\right]$$

$$= \tfrac{1}{2}\times\cdot91503 = \cdot45752$$

thus $\dfrac{A}{2} = 70^\circ 46' 30''$

and $A = \underline{141^\circ 33'}$.

Exercises 25.—On the Solution of Spherical Triangles.

(4-figure log tables only have been used in the solution of these problems.)

In Nos. 1 to 6, solve the spherical triangle ABC, when

1. $a = 50^\circ$ $C = 90^\circ$ $b = 32^\circ 17'$.

2. $C = 90^\circ$ $a = 45^\circ 43'$ $A = 61^\circ 15'$.

3. $a = 72^\circ 14'$ $b = 43^\circ 47'$ $c = 29^\circ 33'$. Find also the value of B by the graphic method explained on p. 368.

4. $b = 52^\circ 5'$ $a = 58^\circ 25'$ $C = 64^\circ$.

5. $b = 27^\circ 13'$ $c = 51^\circ 18'$ $C = 85^\circ 9'$ and the spherical excess is $2^\circ 14'$.

6. $c = 79^\circ 49'$ $b = 28^\circ 5'$ $B = 15^\circ 18'$

7. If the sun's altitude is $17^\circ 58'$, its declination is $18^\circ 16'$ N., and its azimuth is N. $79^\circ 56'$ W., find the latitude of the place of observation.

8. The spherical excess of a triangle on the earth's surface is $1^\circ 15'$: taking the earth as a sphere of radius 3,960 miles, find the area of the triangle in square miles.

9. Given that the azimuth of the sun is 10°, and its zenith distance is $24^\circ 50'$ when its declination is $22^\circ 15'$, find the latitude of the place and also the hour angle.

CHAPTER XIII

MATHEMATICAL PROBABILITY AND THEOREM OF LEAST SQUARES

WHEN extremely accurate results are desired, these results being derived from a series of observations, the possibility of error in each or all of the observations must be considered. The correct result, or what is termed "the most probable result," is usually found by combining the mean of the observations with "the probable error of the mean." The work that is to follow is concerned primarily with the establishment of a rule enabling us to find this probable error; and as a preliminary investigation, a few simple rules of probability will be discussed.

Supposing that an event is likely to happen 5 times and to fail 7 times, then the probability that it will happen on any specified occasion is $\frac{5}{12}$, whilst its probability of failing is $\frac{7}{12}$, because, considered over a great range, it only happens 5 times out of 12. It is important to note the significance of the phrase "*considered over a great range*"; we could not say with truth that the event was bound to happen 5 times out of the first 12, 10 times out of the first 24, and so on; it might be doubtful whether it would happen 50 times out of 120. If, however, say, 12,000 opportunities offered, it would be fairly correct to say that the happenings would be 5,000 and the failures 7,000, for when a large number of occasions were considered, all "freaks" would be eliminated.

To take another illustration:—the probability that a man will score 90 per cent. of the full score or over on a target is $\frac{6}{11}$ indicates that he is rather more likely to score 90 per cent. than not (in the proportion 6 to 5) if he fires a great number of shots.

In general terms, if an event may happen in a ways and fail in b ways, and all these are equally likely to occur, then the probability of its happening is $\frac{a}{a+b}$, and of its failing $\frac{b}{a+b}$; and if

$a=b$, then it is as likely to happen as not, *i. e.*, its probability of either happening or failing is $\frac{1}{2}$.

Note. $\dfrac{\text{Probability of happening}}{\text{Probability of failing}}=\dfrac{a}{a+b}\times\dfrac{a+b}{b}=\dfrac{a}{b}$,

i. e., the odds are a to b for the event, or b to a against it, the first form being used if $a>b$ and *vice versa*.

E.g., if the odds are 10 to 1 against an event, the probability of its happening $=\dfrac{1}{1+10}=\dfrac{1}{11}$; or it will probably happen once only out of eleven attempts.

Exclusive Events.—Let us now consider the case of two exclusive events, viz., the case in which the happenings do not concur.

Suppose the probability of the happening of the first event $=\dfrac{a}{a+b}$ and the probability of the happening of the second event $=\dfrac{A}{A+B}$. Then for purposes of comparison each of these fractions may be expressed with the same denominator: if this common denominator is c, write the fractions as $\dfrac{a_1}{c}$ and $\dfrac{a_2}{c}$ respectively.

Now out of c equally likely ways the first event may happen in a_1 ways and the second in a_2 ways, and since the two events are exclusive, *i. e.*, the happenings of the one do not coincide with the happenings of the other, the two events together may happen in a_1+a_2 ways.

Hence the probability that one or the other will happen is $\dfrac{a_1+a_2}{c}$, which may be written in the form $\dfrac{a_1}{c}+\dfrac{a_2}{c}$, *i. e.*, as the sum of the separate probabilities.

E. g., suppose that one event happens once out of 8 times, and a second event happens three times out of 17, and that there is no possibility of the two events happening together; then, the common denominator of 8 and 17 being 136, the first event happens 17 times out of 136 and the second event happens 24 times out of 136, and hence, either the one or the other happens 41 times out of each 136.

Probability of the Happening together of Two Independent Events.—Suppose that one event is likely to happen once out of every 6 times, whilst another is likely to happen twice out of every 17 times; then the probability that the two will happen together must be smaller than the probability of the

happening of either—in fact, it must be the product of the separate probabilities; *i. e.*, the probability of the two events happening together $=\frac{1}{6}\times\frac{2}{17}=\frac{2}{102}$ or $\frac{1}{51}$; or out of every 10,200 times the first will probably happen 1,700 times, the second will probably happen 1,200 times, whilst the two would happen *together* 200 times only.

Probability of Error.—Bearing in mind these fundamental theorems, we can proceed to a study of the question of probability of error; making particular reference to its application in precision surveying.

It will be admitted that, for any well-made series of observations, the following assumptions may be regarded as reasonable:—

(1) That small errors are more likely to occur frequently than large errors, and hence extremely large errors never occur.

(2) That positive and negative errors are equally likely, *i. e.*, we are as likely to give a result that is ·001 too high as one that is ·001 too low.

Hence the probability of the occurrence of an error of a given magnitude, which is denoted by

$$\frac{\text{the number of errors of that magnitude}}{\text{total number of errors}}$$

depends in some way upon the magnitude of that error. Our first idea, therefore, might be that the probability of the occurrence of an error of magnitude x could be expressed as some linear function of x. It will be seen, however, that this is not in accordance with assumption (2); for assumption (2) demands that if a curve be plotted, the ordinates showing probabilities and the abscissæ indicating errors, it must be symmetrical about the y axis. The function must therefore be of an even power of x, and taking the simplest even power we say that y (the probability of occurrence of an error of magnitude x) $=f(x^2)$.

Now, from assumption (1) we note that the coefficient of x^2 must be negative, because y must decrease as x increases.

The probability of an error of magnitude x being included in the range x to $x+\delta x$ must thus depend on x^2, and also on the range δx; hence it would be reasonable to say that it $=f(x^2)\delta x$, because the greater the range the more is the chance of happening increased. Therefore, the probability that an error of magnitude x falls

between any assigned limits, $-a$ and $+a$, must be the sum of the probability $f(x^2)\delta x$ extended over the range $-a$ to $+a$,

i. e., $$P = \int_{-a}^{a} f(x^2)dx$$

this being the probability that the error does not exceed a.

Hence the probability that the error may have any value whatever (*i. e.*, the probability is 1) must be expressed by

$$\int_{-\infty}^{+\infty} f(x^2)dx,$$

for the range is unlimited, so that

$$\int_{-\infty}^{+\infty} f(x^2)dx \text{ must} = 1.$$

It has been proved by Lord Kelvin that $f(x^2)$ must be such that

$$f(x^2) \times f(y^2) = f(x^2+y^2)$$

and since $$e^{x^2} \times e^{y^2} = e^{x^2+y^2}$$

and $$e^{kx^2} \times e^{ky^2} = e^{k(x^2+y^2)}$$

this condition will be satisfied if

$$f(x^2) = Ae^{kx^2} \text{ or } Ae^{-\frac{x^2}{h^2}}$$

the minus sign being inserted in accordance with assumption 1 on p. 372; and the coefficient k being written as $\frac{1}{h^2}$ for the reason that is explained later.

A value can now be found for the constant A.

It is known that $$\int_{-\infty}^{+\infty} f(x^2)dx = 1,$$

hence $$A\int_{-\infty}^{+\infty} e^{-\frac{x^2}{h^2}}dx = 1.$$

Now it has already been proved (see p. 163) that

$$\int_{0}^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

and $$\int_{0}^{\infty} e^{-\frac{x^2}{h^2}}dx = \frac{h\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{h^2}}dx = 2\int_{0}^{\infty} e^{-\frac{x^2}{h^2}}dx = h\sqrt{\pi}$$

hence $$A\times h\sqrt{\pi}=1,$$

or $$A=\frac{1}{h\sqrt{\pi}}$$

Thus $$y=f(x^2)=\frac{1}{h\sqrt{\pi}}e^{-\frac{x^2}{h^2}}$$

the law being known as the Normal Error Law.

The curve representing this equation is called the probability curve and also Gauss's Error curve. Two such curves are plotted in Fig. 137, to show the effect of the variation of the parameter h. In the one case $h=\cdot 2$, and for the second curve $h=\cdot 5$; and it will be noticed on comparing the curves that for the smaller value of h the probability of the occurrence of *small errors* is greater, *i.e.*, the set of observations for which $h=\cdot 2$ would be more nearly correct than that for which $h=\cdot 5$.

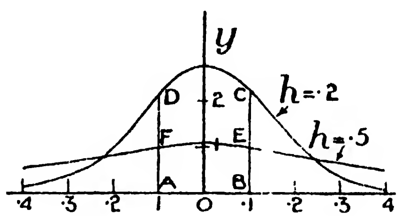


FIG. 137.

It will be seen that the curve is in agreement with the axioms stated on p. 372; for the probability of error is greatest when the error is least, the probability of a large error is very small, and there is as much likelihood of an error of $+\cdot 2$, say, as of $-\cdot 2$.

The probability that the error does not exceed $\cdot 1$ is given by the area ABCD in the one case, and ABEF in the other.

Theorem of Least Squares.—If a number of observations are made upon a quantity, and the errors in each of these noted, *i.e.*, as nearly as can be estimated; then from a knowledge of these errors it is possible to find the most probable or likely value of the quantity.

Let n observations be taken and let the errors be $x_1\ x_2\ \ldots\ x_n$: also suppose that all the measurements are equally good, *i.e.*, the "fineness" of reading is the same throughout; h in the formulæ above being a measure of the fineness.

The probability of the error x_1 being within a certain range δx

will be the probability of an error of magnitude x_1 multiplied by the range δx, *i. e.*,

$$P_1 = \delta x \times y_1$$

$$= \delta x \times \frac{1}{h\sqrt{\pi}} e^{-\frac{x_1^2}{h^2}}$$

and for error x_2 $$P_2 = \delta x \times \frac{1}{h\sqrt{\pi}} e^{-\frac{x_2^2}{h^2}},$$ and so on.

Now x_1 x_2 etc. are quite independent, so that the probability of all the errors falling within the range δx will be the product of the separate probabilities, *i. e.*,

$$P = P_1 \times P_2 \times \ldots P_n$$

$$= \frac{\delta x}{h\sqrt{\pi}} e^{-\frac{x_1^2}{h^2}} \times \frac{\delta x}{h\sqrt{\pi}} e^{-\frac{x_2^2}{h^2}} \times \quad ..$$

$$= \frac{(\delta x)^n}{(h\sqrt{\pi})^n} \left\{ e^{-\left(\frac{x_1^2}{h^2} + \frac{x_2^2}{h^2} + \frac{x_3^2}{h^2} + \ldots\right)} \right\}$$

$$= K e^{-\frac{1}{h^2}(\Sigma x^2)}.$$

We have thus obtained an expression which gives us the probability of all the errors falling within a certain range. We might say that this range was ·1, for instance, or ·05. Evidently if all the errors were kept within the range $-$·05 to $+$·05 the calculated result would be a nearer approximation to the truth than if the range were double the amount stated.

Our object then is to find when the probability of a small error (δx may be reduced as we please) is greatest, *i. e.*, to find when P is a maximum.

Now $$P = K e^{-\frac{1}{h^2}(\Sigma x^2)} = \frac{K}{e^{\frac{1}{h^2}(\Sigma x^2)}}$$

and the smaller the denominator is made, the larger will P become. But the only variable in the denominator is Σx^2, and hence, in order that P may have its maximum value, Σx^2 must be the least possible. Hence the most probable value of the quantity to be determined is that which makes the sum of the squares of the errors the least.

The fact can now be established that the arithmetic mean (A.M.) of the observed values is the most probable value of the quantity.

Thus, if n observations are made,

let $a_1\ a_2\ a_3 \ldots a_n$ be the respective observations

a the A.M. of these values

$\bar{a}$ the value most probably correct

then $(a_1 - \bar{a})$ $(a_2 - \bar{a})$ etc. are known as *residual errors.*

Now the probability of making this system of errors

$$= \mathrm{P}$$

$$= \mathrm{A}e^{-\frac{1}{h^2}[\Sigma(a_1-\bar{a})^2]}$$

or $$\mathrm{P} = \mathrm{A}e^{-\frac{1}{h^2}(a_1^2+\bar{a}^2-2a_1\bar{a}+a_2^2+\bar{a}^2-2a_2\bar{a}+\ldots)}$$

$$= \mathrm{A}e^{-\frac{1}{h^2}[\Sigma a_1^2+n\bar{a}^2-2\bar{a}(\Sigma a_1)]}$$

To differentiate P with regard to $\bar{a}$, put $u = \Sigma a_1^2 + n\bar{a}^2 - 2\bar{a}\Sigma a_1$

so that $$\frac{du}{d\bar{a}} = 0 + 2\bar{a}n - 2\Sigma a_1.$$

Then $$\mathrm{P} = \mathrm{A}e^{-\frac{u}{h^2}}$$

$$\frac{d\mathrm{P}}{d\bar{a}} = \frac{d\mathrm{P}}{du} \times \frac{du}{d\bar{a}}$$

$$= -\frac{\mathrm{A}}{h^2}e^{-\frac{u}{h^2}} \times (2\bar{a}n - 2\Sigma a_1)$$

and $$\frac{d\mathrm{P}}{d\bar{a}} = 0 \text{ if } 2\bar{a}n - 2\Sigma a_1 = 0$$

$$\text{or if } \bar{a} = \frac{1}{n}\Sigma a_1$$

but $\frac{1}{n}\Sigma a_1 = a =$ A.M. of the observations

and hence $\bar{a} = a$

or **the most probable value is the A.M. of the observations.**

Again, if x is the error of the A.M. and $x_1\ x_2\ x_3$ etc. are the respective errors of the observations,

$$x = \frac{1}{n}(x_1 + x_2 + x_3 + \ldots x_n).$$

By squaring

$$x^2 = \frac{1}{n^2}(x_1^2 + x_2^2 + x_3^2 + \ldots x_n^2 + 2x_1x_2 + 2x_1x_3 + \ldots + 2x_2x_3 + \ldots)$$

$$= \frac{1}{n^2}(\Sigma x_1^2) + \frac{2}{n^2}(\Sigma x_1x_2)$$

then, since it is assumed that all the observations are equally good, and that positive and negative errors are equally likely to occur,

$$x_1^2 = x_2^2 = x_3^2 = \ldots = \mu^2 \text{ and } \Sigma x_1 x_2 = 0,$$

for all the errors are small and their products, two at a time, are still smaller.

Also $$x^2 = \frac{1}{n^2}(n\mu^2) = \frac{\mu^2}{n}$$

or $$x = \frac{\mu}{\sqrt{n}}$$

or the probable error of the A.M. $= \dfrac{\text{probable error of a single observation}}{\sqrt{n}}$,

and thus, other things being equal, the possibility of a large error in the final result is greatly reduced by taking a great number of observations. Also in a set of well-made observations, if a sufficient number are made, the arithmetic mean cannot differ from any of the observations to any very great extent, and accordingly the residual errors and the actual errors are very nearly alike.

We are now in a position to summarise the results of the investigation so far as we have pursued it; thus

(*a*) The arithmetic mean of the series of observations, which are supposed to have been made with equal care, is the most probable value of the quantity.

(*b*) The sum of the squares of the residual errors must be the least.

(*c*) The probable error of the A.M. is equal to the probable error of a single observation divided by the square root of the number of observations.

Example 1.—Seven observations of a certain quantity, all made with equal care, were 12, 11, 14, 12, 11·2, 11·7, and 12·1.

Find the most probable value of the quantity.

The most probable value = A.M. $= \frac{84}{7} = \underline{12}$,

and it can readily be shown, by actual calculation, that this value makes the sum of the squares of the residual errors the least.

The residual errors are

(12 − 12), (11 − 12), (14 − 12), (12 − 12), (11·2 − 12), (11·7 − 12) and (12·1 − 12) or 0, −1, 2, 0, −·8, −·3, ·1

and Σ (squares of residual errors) = 0 + 1 + 4 + 0 + ·64 + ·09 + ·01
= 5·74.

To test whether this is the least, let us suppose that the most probable value is 11·5; then the residual errors are: ·5, $-$·5, 2·5, ·5, $-$·3, ·2 and ·6 respectively.

Σ squares $= \cdot 25 + \cdot 25 + 6\cdot 25 + \cdot 25 + \cdot 09 + \cdot 04 + \cdot 36 = 7\cdot 49$.

Similarly, if we assume, say, 12·2 as the most probable value,

$$\begin{aligned}\Sigma\ (\text{residual error})^2 &= (\cdot 2)^2 + (1\cdot 2)^2 + (1\cdot 8)^2 + (\cdot 2)^2 + (1)^2 + (\cdot 5)^2 + (\cdot 1)^2 \\ &= \cdot 04 + 1\cdot 44 + 3\cdot 24 + \cdot 04 + 1 + \cdot 25 + \cdot 01 \\ &= 6\cdot 02\end{aligned}$$

both of which totals exceed 5·74.

To find the Probable Error of the Arithmetic Mean.—Let $r =$ the probable error of any one of the observations; then if this is an "*average*" error, *i. e.*, if errors greater are as likely to occur as errors smaller, the probability that the error is less than r is $\frac{1}{2}$.

Now, the probability that an error lies within the range $-r$ to $+r$

$$\text{is} \qquad \frac{1}{h\sqrt{\pi}}\int_{-r}^{+r} e^{-\frac{r^2}{h^2}}\,dr = \frac{2}{h\sqrt{\pi}}\int_0^r e^{-\frac{r^2}{h^2}}\,dr$$

$$= \frac{2h}{h\sqrt{\pi}}\int_0^{\frac{r}{h}} e^{-\frac{r^2}{h^2}}\,d\left(\frac{r}{h}\right)$$

$\left\{\text{for } dr = hd\left(\frac{r}{h}\right) \text{ and the limits are now those for } \frac{r}{h} \text{ and not those for } r\right\}$.

There must be some connection between the amount of error and the fineness of measurement, *i. e.*, between r and h, and this we must now find.

If $X = \frac{r}{h}$

$$\frac{2}{\sqrt{\pi}}\int_0^{\frac{r}{h}} e^{-\frac{r^2}{h^2}}\,d\left(\frac{r}{h}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\frac{r}{h}} e^{-X^2}\,dX$$

and we see from the above statement that the value of this integral is to be $\frac{1}{2}$.

Now $$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

and thus $$e^{-X^2} = 1 - X^2 + \frac{X^4}{2} - \frac{X^6}{6} + \ldots$$

and if X^2 is small we may perform the integration by way of expansion in series: if X^2 is not small the value of the integral would be read from probability tables which give the values of the integral $\frac{2}{\sqrt{\pi}}\int_0^X e^{-X^2}\,dX$: these tables being given in the *Transactions of the Royal Society of Edinburgh*, Vol. xxxix. For the present

application of the integral, however, X is a small quantity, and a sufficiently correct result is obtained by expanding in a series and calculating from a few terms in this series.

Thus

$$\frac{2}{\sqrt{\pi}}\int_0^{\frac{r}{h}} e^{-X^2}dX = \frac{2}{\sqrt{\pi}}\left(\int_0^{\frac{r}{h}}(1-X^2+\frac{X^4}{2}-\frac{X^6}{6}+\ldots)dX\right)$$

$$= \frac{2}{\sqrt{\pi}}\left(X-\frac{X^3}{3}+\frac{X^5}{10}-\frac{X^7}{42}+\right)_0^{\frac{r}{h}}$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{r}{h}-\frac{r^3}{3h^3}+\frac{r^5}{10h^5}-\frac{r^7}{42h^7}+\ldots\right).$$

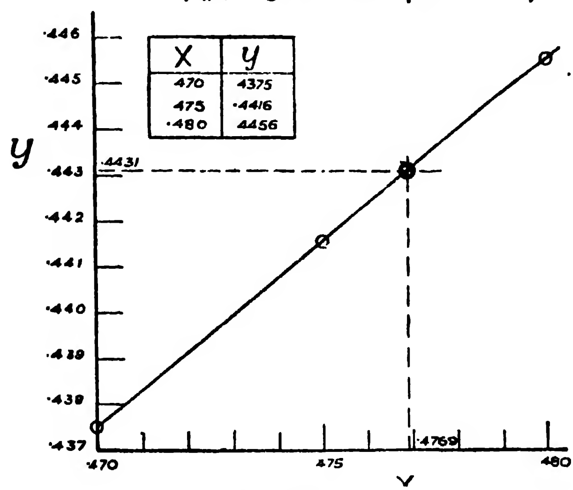


FIG. 138.

Hence $$\frac{1}{2} = \frac{2}{\sqrt{\pi}}\left(\frac{r}{h}-\frac{r^3}{3h^3}+\frac{r^5}{10h^5}-\frac{r^7}{42h^7}+\ldots\right)$$

and this equation may be written

$$\frac{\sqrt{\pi}}{4} = \left(\frac{r}{h}-\frac{r^3}{3h^3}+\frac{r^5}{10h^5}-\frac{r^7}{42h^7}+\ldots\right)$$

or if for $\frac{r}{h}$ we again write X

$$\cdot 4431 = X-\frac{X^3}{3}+\frac{X^5}{10}-\frac{X^7}{42} + \text{terms which are very small.}$$

By selecting values of X and plotting, the solution of this equation is found, the final plotting being represented in Fig. 138, where it is seen that the solution is $X = \cdot 4769$.

Thus $$\frac{r}{h} = \cdot 4769 \text{ or } r = \cdot 4769h.$$

[If solved to a greater degree of accuracy, the value of $\frac{r}{h}$ is found to be $\cdot 47694h$, and this figure will be used in the work that follows.]

Again, if n equally good measurements have been made, each will have what is termed a *weight* of unity, *i. e.*, none is better or worse than any other, and when working towards the result to be deduced from the measurements, equal consideration must be paid to each measurement; also the A.M. is said to have a weight of n since on the average n observations of equal weight must be made to give a result as true as the A.M.

Knowing that $$r_m = \frac{r}{\sqrt{n}}$$

where r_m = probable error of the A.M.

and r = probable error of any observation

and also $$\frac{\text{weight of A.M.}}{\text{weight of one observation}} = \frac{n}{1}$$

which we can write as $\frac{w_m}{w} = \frac{n}{1}$

we can link up w_m and w with r_m and r,

for $$\frac{r_m^2}{r^2} = \frac{1}{n} = \frac{w}{w_m}.$$

or the weight varies inversely as the square of the probable error.

Thus the determination of the probable error, whilst a useful guide to the accuracy of the one set of observations, is more useful in fixing the relative weights that must be given to different sets of observations.

Thus, if three sets of observations have been made on a certain length with the results that the probable errors of the A.M. are 1·423, ·917, and 1·614 respectively; then the weights to be given to these sets are

$$\frac{1}{(1\cdot 423)^2} \quad \frac{1}{(\cdot 917)^2} \quad \frac{1}{(1\cdot 614)^2} \quad \text{respectively}$$

or $$\cdot 494 \qquad 1\cdot 19 \qquad \cdot 384.$$

Then in assessing for the final result, by far the most reliance would be placed on the second set of observations, less on the first, and least on the third set; this fact being well illustrated by

Fig. 139, the resultant weight being nearer to the weight 1·19 than to either of the other weights.

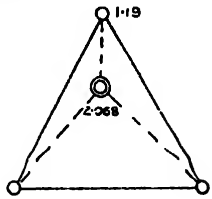


FIG. 139.

To return to the object of this paragraph :—

If x_1 x_2 x_3 . . . are the actual errors of observation, then the probability that each falls within the small range δx is

$$P_1 = \frac{1}{h\sqrt{\pi}} e^{-\left(\frac{x_1}{h}\right)^2} \delta x \qquad P_2 = \frac{1}{h\sqrt{\pi}} e^{-\left(\frac{x_2}{h}\right)^2} (\delta x) \text{ etc.,}$$

and the probability that they all fall within this range at the same time will be less than either of the separate probabilities; it will actually be the product of these.

Thus $\quad P = P_1 \times P_2 \times \ldots\ldots$

$$= \frac{1}{h\sqrt{\pi}} e^{-\left(\frac{x_1}{h}\right)^2} \delta x \times \frac{1}{h\sqrt{\pi}} e^{-\left(\frac{x_2}{h}\right)^2} \delta x \times \ldots\ldots$$

$$= \frac{(\delta x)^n}{h^n \pi^{\frac{n}{2}}} e^{-\frac{1}{h^2}(x_1^2 + x_2^2 + \ldots\ldots)}$$

We wish to find for what value of h P has its greatest value: hence differentiate P with respect to h.

$$P = \frac{K}{h^n} \times e^{-\frac{1}{h^2}(\Sigma x_1^2)} \quad \text{where} \quad K = \frac{(\delta x)^n}{\pi^{\frac{n}{2}}}$$

$$= u \times v$$

$$u = \frac{K}{h^n} \qquad \text{and} \qquad \frac{du}{dh} = K \times -nh^{-n-1} = -nKh^{-(n+1)}$$

$$v = e^{-\frac{1}{h^2}(\Sigma x_1^2)} \quad \text{or if} \quad w = \frac{1}{h^2}(\Sigma x_1^2) \qquad v = e^{-w}$$

and thus $\qquad \dfrac{dw}{dh} = (\Sigma x_1^2) \times -2h^{-3}$

Also $\qquad \dfrac{dv}{dh} = \dfrac{dv}{dw} \times \dfrac{dw}{dh}$

$$= \frac{de^{-w}}{dw} \times \frac{dw}{dh} = -e^{-\frac{1}{h^2}(\Sigma x_1^2)} \times -\frac{2(\Sigma x_1^2)}{h^3}$$

Then

$$\frac{dP}{dh}=v\frac{du}{dh}+u\frac{dv}{dh}$$

$$=\left(e^{-\frac{1}{h^2}(\Sigma x_1^2)}\times-nKh^{-(n+1)}\right)+\left(\frac{K}{h^n}\times 2e^{-\frac{1}{h^2}(\Sigma x_1^2)}\times\frac{(\Sigma x_1)^2}{h^3}\right)$$

$$=Ke^{-\frac{1}{h^2}(\Sigma x_1^2)}\left[-nh^{-(n+1)}+2h^{-(n+3)}(\Sigma x_1^2)\right]$$

and $\frac{dP}{dh}=0$ if $nh^{-(n+1)}=2h^{-(n+3)}(\Sigma x_1^2)$

i.e., if $h^2=\frac{2\Sigma x_1^2}{n}$

or $h=1{\cdot}414\sqrt{\frac{\Sigma x_1^2}{n}}$

Now it has already been proved that $r=\cdot 47694h$

so that $r=\cdot 47694\sqrt{2}\sqrt{\frac{(\Sigma x_1^2)}{n}}$

$$=\cdot 6745\sqrt{\frac{(\Sigma x_1^2)}{n}}.$$

Also we have previously stated that the sum of the squares of the actual errors differs very little from the sum of the squares of residual errors; this being true if a great number of observations are taken. The difference in the two sums may be expressed rather more accurately by the relation

$$\Sigma x_1^2=\frac{n}{n-1}\Sigma\text{ (residual error)}^2.$$

Hence if for Σ(residual error)^2 we write $\Sigma(r_e^2)$

$$r=\cdot 6745\sqrt{\frac{(\Sigma r_e^2)}{n-1}}$$

and $r_m=\frac{r}{\sqrt{n}}=\cdot 6745\sqrt{\frac{(\Sigma r_e^2)}{n(n-1)}}.$

Applying these results to *Example* 1 on p. 377,

$\Sigma r_e^2=5{\cdot}74$

$n=7$

then $r=\cdot 6745\sqrt{\frac{5{\cdot}74}{6}}=\cdot 656$

$$r_m=\cdot 6745\sqrt{\frac{5{\cdot}74}{7\times 6}}=\cdot 2475$$

$$\text{also} \quad h = \sqrt{\frac{2n\Sigma r_e^2}{(n-1)\times n}} = \sqrt{\frac{2\times 5\cdot 74}{6}} = 1\cdot 38$$

i. e., h has a very high value; and this would be expected, for the "*fineness*" of reading, as judged by the results, is not at all good (one error being as much as 2 in 12).

Example 2.—In a chain survey four measurements of a base line gave 867·35, 867·51, 867·28 and 867·62 links respectively. Find the best length and the probable error in this length.

The best result is the A.M. of these, *i. e.*,

$$\frac{867\cdot 35+867\cdot 51+867\cdot 28+867\cdot 62}{4}$$

$$= 867\cdot 44 \text{ links}$$

and whilst this is the best result it contains a probable error.

Probable error in A.M.

$$= r_m = \cdot 6745\sqrt{\frac{\Sigma r_e^2}{n(n-1)}}$$

$$= \cdot 6745\sqrt{\frac{(-\cdot 09)^2+(\cdot 07)^2+(-\cdot 16)^2+(\cdot 18)^2}{4\times 3}}$$

$$= \cdot 6745\sqrt{\frac{\cdot 071}{12}}$$

$$= \cdot 0517$$

i. e., the base line measurement (867·44) is subject to an error of ·0517 link, and as this result could not be bettered it would be unnecessary to repeat this portion of the survey.

The probable error in any one observation would be

$$r = \cdot 6745\sqrt{\frac{\cdot 071}{3}} = \cdot 103,$$

so that there is a decided gain in accuracy obtained by increasing the number of observations. (Cf. "repetition," when working with the theodolite.)

It is of interest to find h for this example.

$$h = \sqrt{\frac{2\times \cdot 071}{3}} = \cdot 2176$$

and as this is a small quantity we are confirmed in our conclusion that the observations were well made.

Example 3.—The mean values of the three angles of a spherical triangle were calculated from the actual observations to be 75° 40′ 21·6″, 39° 11′ 47·3″, and 65° 7′ 56·2″; and these values were subject to probable errors 2·9″, 3·6″, and 4·3″ respectively. From a knowledge

of the area of the triangle, the spherical excess of the triangle was found to be $3 \cdot 3''$. Make the necessary adjustments to the angles to satisfy this condition.

The actual spherical excess

$$= (75^\circ 40' 21 \cdot 6'' + 39^\circ 11' 47 \cdot 3'' + 65^\circ 7' 56 \cdot 2'') - 180^\circ$$
$$= 5 \cdot 1''.$$

There is thus $(5 \cdot 1 - 3 \cdot 3)$ to be divided among the angles, according to the respective weights; and these weights are in the proportion

$$2 \cdot 9^2 \quad 3 \cdot 6^2 \quad 4 \cdot 3^2$$

or

$$8 \cdot 41 \quad 12 \cdot 96 \quad 18 \cdot 49,$$

the sum of the weights being $39 \cdot 86$.

Hence the corrections to be applied are $\frac{8 \cdot 41}{39 \cdot 86} \times 1 \cdot 8$, $\frac{12 \cdot 96}{39 \cdot 86} \times 1 \cdot 8$, and $\frac{18 \cdot 49}{39 \cdot 86} \times 1 \cdot 8$ to the respective angles; all these corrections being subtracted, since the observed angles give a spherical excess greater than should actually be the case.

These corrections are $\cdot 380$, $\cdot 585$, and $\cdot 835$.

Hence the true angles are $(75^\circ 40' 21 \cdot 6'' - \cdot 38'')$, $(39^\circ 11' 47 \cdot 3'' - 59'')$ and $(65^\circ 7' 56 \cdot 2'' - \cdot 84'')$,

or $\underline{75^\circ 40' 21 \cdot 22'', \; 39^\circ 11' 46 \cdot 71'' \text{ and } 65^\circ 7' 55 \cdot 36''}$.

Example 4.—Measurements of an angle in a traverse survey were made by two different observers, with the following results :—

Readings by A.	Readings by B.
76° 50′ 20″	76° 50′ 55″
76° 50′ 50″	76° 50′ 35″
76° 50′ 30″	76° 51′ 15″
76° 51′ 10″	76° 51′ 20″
76° 50′ 30″	76° 51′ 0″
76° 51′ 0″	76° 50′ 45″
76° 50′ 40″	76° 50′ 25″
76° 50′ 30″	76° 50′ 40″

Compare the two results from the point of accuracy, and find the most probable value of the angle.

We must first find the arithmetic mean of each set of observations, and then, by subtracting this from each reading, we determine the residual errors.

The A.M. of set A $= 76^\circ 50' 41 \cdot 25''$
and A.M. of set B $= 76^\circ 50' 51 \cdot 88''$.

Since the differences are of seconds only, we need not concern ourselves for the present with the degrees and minutes; and thus the table of residual errors and their squares becomes

A		B	
Residual Error.	(Residual Error)^2.	Residual Error.	(Residual Error)^2.
$-21{\cdot}25$	$451{\cdot}4$	$+\ 3{\cdot}12$	$9{\cdot}7$
$+\ 8{\cdot}75$	$76{\cdot}6$	$-16{\cdot}88$	$284{\cdot}9$
$-11{\cdot}25$	$126{\cdot}6$	$+23{\cdot}12$	$534{\cdot}6$
$+28{\cdot}75$	$826{\cdot}8$	$+28{\cdot}12$	$790{\cdot}7$
$-11{\cdot}25$	$126{\cdot}6$	$+\ 8{\cdot}12$	$66{\cdot}0$
$+18{\cdot}75$	$351{\cdot}6$	$-\ 6{\cdot}88$	$47{\cdot}3$
$-\ 1{\cdot}25$	$1{\cdot}6$	$-26{\cdot}88$	$722{\cdot}4$
$-11{\cdot}25$	$126{\cdot}6$	$-11{\cdot}88$	$141{\cdot}2$.
sum 0	$2087{\cdot}8$	0	$2596{\cdot}8$

In case A $\qquad r_m = {\cdot}6745\sqrt{\dfrac{2087{\cdot}8}{8\times 7}} = 4{\cdot}119.$

In case B $\qquad r_m = {\cdot}6745\sqrt{\dfrac{2596{\cdot}8}{8\times 7}} = 4{\cdot}594.$

Then the $\dfrac{\text{weight of observations by A}}{\text{weight of observations by B}} = \dfrac{(4{\cdot}594)^2}{(4{\cdot}119)^2} = \dfrac{1{\cdot}244}{1}$

Thus A's readings can be relied on before those of B; the former being roughly $1\frac{1}{4}$ times as good as the latter.

The most probable value of the angle, taking into account the two sets of readings, will be obtained by the calculation of the "weighted mean," *i. e.*, the mean of the two arithmetic means already found, determined with due regard to the respective weights to be given to A's readings and B's readings.

Dealing only with the seconds, the most probable value

$$= \frac{(41{\cdot}25\times 1{\cdot}244)+(51{\cdot}88\times 1)}{1+1{\cdot}244}$$

$$= \frac{51{\cdot}31+51{\cdot}88}{2{\cdot}244} = \frac{103{\cdot}19}{2{\cdot}244} = 46 \text{ seconds.}$$

Hence the most probable value of the angle $= \underline{76^\circ\ 50'\ 46''}$.

Application of the Principle of Least Squares to the Determination of Laws.—If corresponding values of two variables

x and y are given, and it is thought that the relation between them is $y = a + bx$ the values of a and b can be found from a plotting of y against x. This method necessitates the careful selection of the best straight line for the plotted points, and, in cases in which the values to be plotted are obtained from experimental results, it is often difficult to decide which is really the best line. The most probable law can, however, be found without any plotting, and with great accuracy, by a process based upon the principle of least squares.

The most probable law is that which makes the sum of the squares of the differences between the values of y calculated from the equation and the given values of y the least. Thus, if $y = a + bx$ is the form of equation desired, the most probable values of a and b are those found from the fact that $\Sigma(y - y_c)^2$ must be the least; y_c being the calculated value of y.

Now $y_c = a + bx$, so that $\Sigma(y - a - bx)^2$ must be a minimum.

Again, $\Sigma(y - a - bx)^2$ depends upon both a and b, so that it is a minimum if

$$\frac{\partial}{\partial a}\Sigma(y - a - bx)^2 = 0 \qquad [b \text{ being constant}]$$

and

$$\frac{\partial}{\partial b}\Sigma(y - a - bx)^2 = 0 \qquad [a \text{ being constant}]$$

$$\frac{\partial}{\partial a}\Sigma(y - a - bx)^2 = 0 \quad \text{if } \frac{\partial\Sigma(y - a - bx)^2}{\partial(y - a - bx)} \times \frac{\partial(y - a - bx)}{\partial a} = 0$$

i. e. if $\quad 2\Sigma(y - a - bx) \times -1 = 0$

if $\quad \Sigma y - na - b\Sigma x = 0$

or if $\quad \Sigma y = na + b\Sigma x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

where n is the number of sets of values and consequently the number of times that a occurs when calculating the differences.

In like manner $\quad \frac{\partial}{\partial b}\Sigma(y - a - bx)^2 = 0$

if $\quad 2\Sigma(y - a - bx) \times -x = 0$

i. e. if $\quad \Sigma xy - a\Sigma x - b\Sigma x^2 = 0$

or $\quad \Sigma xy = a\Sigma x + b\Sigma x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$

so that two equations can be formed from the given values of x and y, and the values of a and b found therefrom.

Example **5.—In a test on two metallic filament lamps connected in parallel the following values of the volts V and the resistance R were obtained :—**

V	54	60	65	70	75	80	85	90	105
R	108	109	114	118	123	127	131	134	146

Find the most probable law connecting R and V in the form

$$R = a + bV.$$

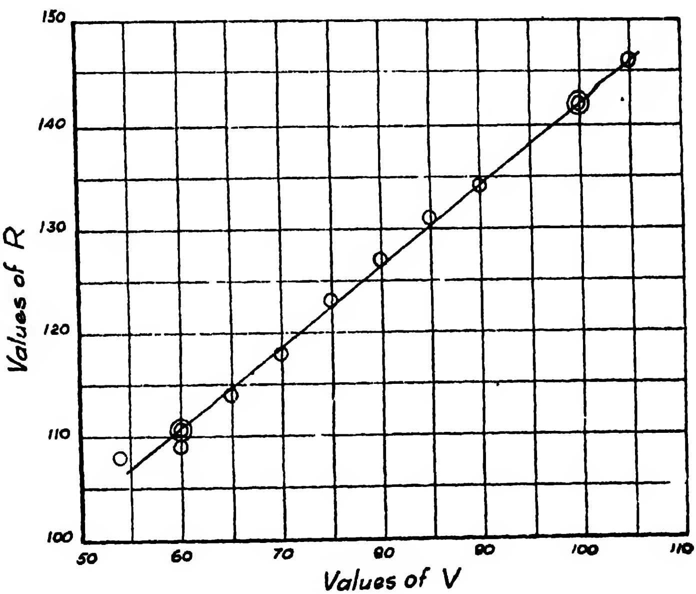


FIG. 140.

In Fig. 140 R is plotted against V and a line is drawn which seems to fit the points well. Selecting two convenient points on the line, and substituting the values in the equation, we have

$$142 = 100b + a$$

$$111 = \ \ 60b + a$$

whence $\quad 31 = 40b \quad$ and $\quad b = \cdot 775$

and $\quad a = 142 - 77{\cdot}5 = 64{\cdot}5$

or the equation obtained from the graph is $R = 64{\cdot}5 + \cdot 775V$.

We cannot be certain that this is the most accurate equation for the values as given; to obtain this we proceed as follows:—

V	54	60	65	70	75	80	85	90	105	$\Sigma V = 684$
R	108	109	114	118	123	127	131	134	146	$\Sigma R = 1110$
VR	5832	6540	7410	8260	9225	10160	11135	12060	15330	$\Sigma VR = 85952$
V^2	2916	3600	4225	4900	5625	6400	7225	8100	11025	$\Sigma V^2 = 54016$

also $n = 9$, hence

$$1110 = 9a + 684b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$85952 = 684a + 54016b \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Multiplying (1) by 76 and subtracting

$$85952 = 684a + 54016b$$
$$84360 = 684a + 51984b$$
$$1592 = 2032b$$

or
$$b = \cdot 7835$$

and, by substitution in equation (1)

$$1110 = 9a + 535\cdot 98$$
$$a = 63\cdot 78.$$

Hence the most probable law for the given values is

$$\underline{R = 63\cdot 78 + \cdot 7835V.}$$

We may test this result by calculating values of R from it, and comparing with the values of R calculated from $R = 64\cdot 5 + \cdot 775V$, thus:—

V	54	60	65	70	75	80	85	90	105	—
R_g	106·35	111	114·875	118·75	122·625	126·5	130·375	134·25	145·875	—
R_c	106·089	110·79	114·7075	118·625	122·5425	126·46	130·3775	134·295	146·0475	—
$R - R_g$	1·65	− 2	− ·875	− ·75	·375	·5	·625	− ·25	·125	$\Sigma = -\cdot 6$
$R - R_c$	1·911	− 1·79	− ·7075	− ·625	·4575	·54	·6225	− ·295	− ·0475	$\Sigma = \cdot 066$
$(R - R_g)^2$	2·7225	4	·7656	·5625	·1406	·25	·3906	·0625	·0156	$\Sigma = 8\cdot 90{,}9$
$(R - R_c)^2$	3·6519	3·2041	·5006	·3906	·2093	·2916	·3875	·0870	·0023	$\Sigma = 8\cdot 7249$

where $R_g = 64\cdot 5 + \cdot 775V$ and $R_c = 63\cdot 78 + \cdot 7835V$.

It is seen that both $\Sigma(R - R_c)$ is less than $(R - R_g)$ and $\Sigma(R - R_c)^2$ is less than $\Sigma(R - R_g)^2$, the differences being small, which indicates

that the choice of the line can be made with reasonable accuracy even in cases where the points are somewhat off the line.

When the values of y and x are connected by an equation of the form $y = a + bx + cx^2$ the points lie on, or about, a curve, and it is more difficult to fit a curve to a series of points than to find the best straight line for them. In cases where great accuracy is desired, it is the better plan to use the method of this paragraph, although the calculation is wearisome.

To find the most probable values of a, b and c, it is necessary to equate to zero the partial derivatives of $\Sigma(y - a - bx - cx^2)^2$ with respect to a, b and c respectively.

By so doing the three equations obtained are:—

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

and consequently Σx, Σx^2, Σx^3, Σx^4, Σy, Σxy and $\Sigma x^2 y$ must be found.

Example 6.—If $y = a + bx + cx^2$ and corresponding values of y and x are

x	0	2	5	10
y	4	7	6·4	− 6

find the most probable values of a, b and c.

The table of values reads :—

x	0	2	5	10	$\Sigma = 17$
y	4	7	6·4	− 6	$\Sigma = 11·4$
xy	0	14	32	− 60	$\Sigma = -14$
x^2	0	4	25	100	$\Sigma = 129$
x^2y	0	28	160	− 600	$\Sigma = -412$
x^3	0	8	125	1000	$\Sigma = 1133$
x^4	0	16	625	10000	$\Sigma = 10641$

from which the equations are formed,

$$\begin{aligned} 11\cdot 4 &= 4a+17b+129c \\ -14 &= 17a+129b+1133c \\ -412 &= 129a+1133b+10641c \end{aligned}$$

and the solutions of these equations are

$$a = 4\cdot 0771,\quad b = 1\cdot 9794,\quad c = -\cdot 2989$$

so that the best law for the given values is

$$\underline{y = 4\cdot 0771 + 1\cdot 9794x - \cdot 2989x^2}.$$

Exercises 26.—On the Calculations of Errors of Measurements.

1. One surveying party measured a certain base line as 6 chains 42·7 links, 6 chains 53·5 links, 6 chains 46·4 links, and 6 chains 41·9 links; and a second party measured the same line as 6 chains 38·4 links, 6 chains 39·7 links, 6 chains 46·9 links, and 6 chains 43 links. State which of the two parties is the more dependable, and find the most probable length of the line.

2. Plot the probability curve $y = \frac{1}{h\sqrt{\pi}}e^{-\frac{x^2}{h^2}}$ the value of h being ·1414, and find the probability that an error lies within the limits $-\cdot 6$ and $+\cdot 6$.

3. The following are the values of the determination of the azimuth of Allen from Sears, Texas, the results of a U.S. Coast and Geodetic Survey; the values of the seconds only being stated after the first reading: 98° 6′ 41·5″, 42·8, 43·4, 43·1, 39·7, 42·7, 41·6, 43·3, 40·0, 45·0, 43·3 and 40·7. Find the A.M., the probable error of a single observation and the probable error of the mean.

4. Find the weighted mean of the following observations: 95·8, 96·9, 97·2, 95·4, 95·7, 97·1, 96·5, 96·7 and 97; the probable errors in the measurements being ·2, ·4, ·1, ·9, ·7, 1·2, ·8, ·3 and 1·5 respectively.

5. Find the most probable relation between the twisting moment T and the angle of twist α for the following case:—

α	0	·34	·67	1	1·35	1·7	2·05
T	0	1200	2400	3600	4800	6000	7200

6. Calculate the probable error r of a single observation, taking the figures in Example 1, p. 377, but using Peters' formula

$$r = \frac{\cdot 8453}{\sqrt{n(n-1)}}\,\Sigma\,\text{[residuals (without reference to sign)]}.$$

ANSWERS TO EXERCISES

Exercises 1, p. 22

3. $E = \text{constant} \times \dfrac{dZ}{dt}$ **4.** $V = RC + L\dfrac{dC}{dt}$

5. $-11 \cdot 03$ cms. per sec. : $1 \cdot 07$ secs. from start **6.** $\cdot 336$ ton

11. $5 \cdot 65$ **12.** Middle of May: middle of October

15. Loading is $\cdot 2$ ton per foot run **16.** $\cdot 966 : \dfrac{d \sin \theta}{d\theta} = \cos \theta$

17. $\cdot 42$ ton per foot **21.** $6 \cdot 3$ **24.** $y = 8x - 11$

Exercises 2, p. 36

1. $4x^3$ **2.** -128 **3.** $-2h$ **4.** $27x^8$ **5.** $-\dfrac{75}{x^6}$

6. $-\dfrac{18 \cdot 75}{x^{1 \cdot 23}}$ **7.** $32 \cdot 6x^{\cdot 72}$ **8.** $\dfrac{\cdot 0086}{x^{\cdot 84}}$ **9.** $31 \cdot 8x^{10 \cdot 68}$ **10.** $-\dfrac{1 \cdot 11}{x^{\frac{1}{9}}}$

11. $-\dfrac{\cdot 982}{x^{2 \cdot 3}}$ **12.** $\cdot 62x^{6 \cdot 3}$ **13.** $45x^2 - 44 \cdot 8x^{1 \cdot 8} + \dfrac{21}{x^4}$

15. $-\dfrac{\cdot 19a^2b^{1 \cdot 75}}{x^{1 \cdot 5}}$ **16.** $-\cdot 347$

17. $\cdot 71v^{3 \cdot 84} + \dfrac{\cdot 84}{v^{4 \cdot 36}} + \dfrac{1 \cdot 29}{v^{4 \cdot 44}} - \dfrac{\cdot 525}{v^{5 \cdot 2}} - \dfrac{12 \cdot 48}{v^{5 \cdot 16}}$ **18.** $10 \cdot 74$ **19.** $\cdot 25p$

20. $\cdot 073$ **21.** $1 - a - \frac{2}{3}bP^{-\frac{1}{3}}$ **22.** $w\left(\dfrac{xy}{l} + \dfrac{xy^2}{l^2} - \dfrac{y^2}{l} - x\right)$

23. $\cdot 289l$ **24.** $\dfrac{w}{yl}(xy + ny^2 - xl)$ **25.** $h = \dfrac{d}{2}\sqrt{1 + \dfrac{l}{d}}$

26. $-2(p + q)$ **27.** $7 \cdot 85$ **28.** $w\left(\dfrac{ry}{l} - x\right)$ **29.** $9 \cdot 6$

30. -7333

Exercises 3, p. 46

1. Sub-normal $= 466$; sub-tangent $= 1$ **2.** $25 \cdot 7$

3. $y = \cdot 0256x$; (x is distance from centre): $\cdot 64$

4. $p = -15 \cdot 65v + 143 \cdot 5$ **5.** $M = \dfrac{W}{2}\left(\dfrac{l}{2} - x\right)$: $S = -\dfrac{W}{2}$: $L = 0$

6. $M = \dfrac{w}{2}\left(\dfrac{l^2}{4} - x^2\right)$: $S = -wx$: $L = -w$

7. $M = W(l - x)$: $S = -W$: $L = 0$

8. Sub-tangent $= \dfrac{81a^3}{5b^3}$: sub-normal $= 5b^3$

9. 3 **10.** $\dfrac{x}{2EI}(wl - wx - 2P)$

Exercises 4, p. 55

1. $-5e^{-5x}$ **2.** $6\cdot15e^{4\cdot1x}$ **3.** $-\frac{189}{e^{7x}}$ **4.** $1\cdot423\,(4\cdot15)^x$

5. $4\cdot33\,(8\cdot72)^{2x}$ **6.** $9e^{9x} - 35e^{-7x}$ **7.** $5\cdot44x^{1\cdot718}$ **8.** $9\cdot7\,(2)^x$

9. $10\cdot25e^{\cdot25x} - \frac{3\cdot04}{e^{\cdot16x}}$ **10.** $16e^{5\cdot1x} - 47\cdot25x^{3\cdot45} + 3\cdot39\,(3\cdot1)^{2x}$

11. $12\cdot6e^{4\cdot2x}$ **12.** $\frac{1}{x}$ **13.** $\frac{15}{5x-4}$ **14.** $\frac{4\cdot343}{x}$

15. $\frac{4ac}{4ax+5b}$ **16.** $-1\cdot8e^{-\cdot2x} - \frac{1}{x} - \frac{2\cdot46}{x^{1\cdot43}}$

17. $\frac{1}{x} + \frac{3}{3x-4\cdot7}$ or $\frac{6x-4\cdot7}{x(3x-4\cdot7)}$ [Use the rule $\log AB = \log A + \log B$]

18. $\frac{5}{5x+4} + \frac{3}{3x-2} + \frac{4}{7-4x}$ **19.** $3e^{3x} + 8\sinh 2x - \frac{\cdot738}{x}$

20. $\frac{2}{x} - \frac{3\cdot5}{x^{1\cdot7}} - 1\cdot057\,(1\cdot8)^x$ **21.** 0 **22.** 1·052 **23.** $\frac{2\cdot84}{\cdot04u - 18}$

24. $\frac{\cdot4343}{t}$ **25.** ·2T **26.** $\frac{C_0R}{L}e^{-\frac{Rt}{L}}$ **27.** $\frac{1\cdot587}{1\cdot8u - 7}$

28. $\frac{5}{4}\sinh\frac{x}{4} : \frac{p}{q}\cosh\frac{y}{q}$ **29.** lrE **30.** $\frac{p_1}{p_b}$ **31.** b

32. 0 **33.** 0 **34.** $\frac{B}{\tau} + C - \frac{D}{\tau^2}$

Exercises 5, p. 62

1. $-5\cdot3\cos(4 - 5\cdot3x)$ **2.** $-16\cdot32\sin 5\cdot1x$ **3.** $\cdot48\sec^2(3x+9)$

4. $\cdot914\cos(\cdot425x - 1\cdot25)$ **5.** $-40\operatorname{cosec}^2 x$ **6.** $-\frac{5\cdot05\sin(\cdot05 - \cdot117x)}{\cos^2(\cdot05 - \cdot117x)}$

7. $gbc\sin(d - gx)$ **8.** $-20\sin 5x - 14\cos(2x-5)$

9. $4\cdot4\cos 8\cdot8x + \cdot8\cos 1\cdot6x$ {Use the rule: $2\sin A\cos B = \sin(A+B) + \sin(A-B)$} **10.** $-6\cdot74\sin 6\cdot2x - 3\cdot04\sin 2\cdot8x$

11. $4\cdot52\,\{(q-p)\sin(px - qx + 2c) + (p+q)\sin(px+qx)\}$

12. $5\sin 2x$. {Use the rule: $\cos 2A = 1 - 2\sin^2 A$} **13.** $-\cdot195\sin 6x$

14. 0 **15.** $5\cdot16x^{\cdot72} - \frac{15\cdot42}{3x - 4\cdot1} - \cdot0273\cos(4\cdot31 - \cdot195x) + 24\,93$

16. $\cdot1056\cos\cdot015x - \cdot0529\sin(6\cdot1 - \cdot23x) + 7\cdot4\sec^2(4x - \cdot07)$

17. Velocity $= 37\cdot7\sin 31\cdot4t - 56\cdot56\cos 31\cdot4t$:
acceleration $= 1184\cos 31\cdot4t + 1777\sin 31\cdot4t$

18. Acceleration $= -\cdot0289s$: S.H.M. **19.** -1166

20. Sine curve (*i.e.*, second derived curve).

21. 0 $\left\{\text{Treat as a constant the portion } \frac{Bl}{8\left(\frac{EI\pi^2}{l^2} - F\right)}\right\}$

22. $\frac{dE}{dt} = 1500p\cos pt + 300p\cos 3pt + 42p\sin pt - 84p\sin 3pt$

Exercises 6, p. 68

1. $2 \cos 2x \,.\, e^{\sin 2x}$ 2. $\dfrac{2}{v}$ 3. $-2 \sin 2t$ 4. $24x^2 . \cos x^3$

5. $3{\cdot}14\,(10x + 7) \sec^2 (5x^2 + 7x - 2)$ 6. $3 \log a \,.\, \cos 3x \,.\, a^{\sin 3x}$

7. $1{\cdot}85x^{\cdot 85} \,.\, e^{x^{1\cdot 85}}$ 8. $\dfrac{\cdot 4343\,(7 - 27x^2)}{3 + 7x - 9x^3}$ 9. $-\dfrac{5 \sin (\log s^5)}{s}$

10. $\operatorname{cosec} x$ 11. $\dfrac{\sec^2 \theta}{\sec^2 \phi \cos a}$ 12. $\dfrac{-\sin \theta}{1 + \cos \theta}$ $\left\{\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx}\right\}$

13. $\left\{\dfrac{d \log h}{d \log V} = \text{slope of curve} = \dfrac{d \log h}{dh} \times \dfrac{dh}{dV} \times \dfrac{dV}{d \log V}\right\}$

15. 1·08 ft. per min.: ·377 ft. per min. 18. ·0332 lb.: $\cdot 0102d$

19. $7 \operatorname{cosec} 7t + 45t^2$ 20. $\dfrac{Ful}{EA}$ 21. 56° 19′ 22. 53° 7′ 24. $\dfrac{144t^2}{6t^3 - 37}$

Exercises 7, p. 73

1. $x\,(2 \sin 3x + 3x \cos 3x)$ 2. $2x^{2\cdot 4}\,(1 + 3{\cdot}4 \log 5x)$

3. $e^{9x}\left(9 \log_{10} 9x + \dfrac{\cdot 4343}{x}\right)$

4. $-\dfrac{4}{x^5 \cos (3{\cdot}1 - 2{\cdot}07x)}\left\{\dfrac{2{\cdot}07}{\cos (3{\cdot}1 - 2{\cdot}07x)} + \dfrac{5 \sin (3{\cdot}1 - 2{\cdot}07x)}{x}\right\}$

5. $-\{2{\cdot}575 \sin (5{\cdot}15x + 4) + \cdot 625 \sin (1{\cdot}25x - 4)\}$ or
$-\{3{\cdot}2 \sin 3{\cdot}2x \cos (1{\cdot}95x + 4) + 1{\cdot}95 \cos 3{\cdot}2x \sin (1{\cdot}95x + 4)\}$

6. $\sec^2 2x\,\{2 \cos (5 - 3x) + 1{\cdot}5 \sin (5 - 3x) \sin 4x\}$ or
$3 \tan 2x \sin (5 - 3x) + 2 \sec^2 2x \cos (5 - 3x)$

7. $12{\cdot}8x^{\cdot 6}\{\cos (3 + 8x) + 2 - 5x \sin (3 + 8x)\}$

8. $27\,(5)^{3x}\left\{4{\cdot}83 \log x + \dfrac{1}{x}\right\}$ 9. $(1 + \log x)\, e^{x \log x}$ 10. $5x^2 e^{4x}$

11. $30e^{5x+2}\,(5x + 4)\,(5x + 2)$ 12. $50{\cdot}4\left(\dfrac{\tan \cdot 125x}{x} + \dfrac{\log x}{8 \cos^2 \cdot 125x}\right)$

13. 0 14. $-5e^{-10t}$ 15. 0 16. $12600 \sin (14t - \cdot 116)$

17. $6t\,\{5 \sin (4 - \cdot 8t) - 2t \cos (4 - \cdot 8t)\}$ 18. $4t^{2\cdot 7}\,(3{\cdot}7 \cos 3t - 3t \sin 3t)$

Exercises 8, p. 78

1. $\dfrac{5x^2\,(3 - 7x)}{e^{7x-5}}$ 2. $\dfrac{7}{\cos (2 - 7x)}\left\{\dfrac{1}{7x - 2} - \log (2 - 7x) \tan (2 - 7x)\right\}$

3. $\dfrac{20}{\sqrt{49 - 16x^2}}$ 4. $-\dfrac{b}{\sqrt{d^4 - b^2x^2}}$ 5. $-\dfrac{5{\cdot}46\,(5)^{2\cdot 2x}}{e^{3x-4}}$

6. $\dfrac{1{\cdot}8}{4^{1\cdot 8x}}(\sinh 1{\cdot}8x - 1{\cdot}386 \cosh 1{\cdot}8x)$ or $-\dfrac{\cdot 9}{4^{1\cdot 8x}}(2{\cdot}386e^{-1\cdot 8x} + \cdot 386e^{1\cdot 8x})$

7. $\dfrac{1}{x^2 + 6x + 15}$ 8. $\dfrac{63x \cos^{-1} 3x - 21\sqrt{1 - 9x^2}}{(1 - 9x^2)^{\frac{3}{2}}}$

9. $\dfrac{wb\,(ab - 2bx + x^2 \cot B)}{2\,(b - x \cot B)^2}$ 10. $\dfrac{1}{(a^2 + x^2)^{\frac{3}{2}}}$

11. $\dfrac{56x^3 - 111lx^2 + 72l^2x - 14l^3}{(3l - 4x)^2}$

12. $\dfrac{e^{\sin(1\cdot 2x + 1\cdot 7)}}{\{\log(8x^2 - 7x + 3)\}^2} \times \left\{1\cdot 2\cos(1\cdot 2x + 1\cdot 7)\log(8x^2 - 7x + 3) + \dfrac{7 - 16x}{8x^2 - 7x + 3}\right\}$

13. $\operatorname{sech}^2 x$

14. $\dfrac{6\cdot 55(\sin\theta - \theta\cos\theta)}{\theta^{\frac{3}{2}}(\theta - \sin\theta)^{\frac{1}{2}}}$

15. $\dfrac{66\cdot 2(\theta - \sin\theta)^{\frac{1}{2}}(2\theta - 3\theta\cos\theta + \sin\theta)}{\theta^{\frac{5}{2}}}$

16. $\dfrac{y^2}{1 - y^4}$

17. $r\omega^2\left[\cos\theta + \dfrac{\sin^4\theta - 2m^2\sin^2\theta + m^2}{(m^2 - \sin^2\theta)^{\frac{3}{2}}}\right] : r\omega^2\left(1 + \dfrac{1}{m}\right)$

18. $\dfrac{d\phi}{dt} = \dfrac{\omega\cos\theta}{\sqrt{m^2 - \sin^2\theta}} : \dfrac{d^2\phi}{dt^2} = \dfrac{\omega^2\sin\theta(1 - m^2)}{(m^2 - \sin^2\theta)^{\frac{3}{2}}}$

19. $\dfrac{w\tan^2\theta - q}{(p - q)\sin^2\theta} : \pm\sqrt{\dfrac{q}{w}}$

20. $\dfrac{w}{2(3l - 2x)^2}(3l^3 + 24x^2l - 18xl^2 - 8x^3)$

21. $\dfrac{5}{t^4}\left\{\dfrac{5t}{5t - 8} - 3\log(5t - 8)\right\}$

Exercises 9, p. 86

1. $\sqrt{r_1^2b^2 + r_2^2a^2}$

2. $7\cdot 52x^5e^{2v} : 75\cdot 2e^{2v}x^3$

3. $\dfrac{5e^{4t}}{5p - 3} - t^2 : 5\cdot 2t^{4\cdot 2} - 2pt + 4e^{4t}.\log(5p - 3)$

4. $10(4 - u)(9 - 4u)(3 + 8u)^2$

6. $8(1\cdot 7 + x)(1\cdot 7 + \cdot 2x)^3$

7. $-\dfrac{3x^2 + 18x + 31}{(x^2 + 6x + 5)^3}$

8. $\dfrac{v}{c\tau}\left(\dfrac{dp}{dt} - \dfrac{p}{\tau}\cdot\dfrac{d\tau}{dt} + \dfrac{p}{v}\dfrac{dv}{dt}\right)$

Exercises 10, p. 102

1. 750
2. 17·1
3. $\dfrac{l}{2} : \dfrac{\omega l^2}{8}$
4. ·577l : ·128Wl
5. ·5
6. − 2·25 : minimum
7. ·278 : maximum
8. maximum at $x = -5$: point of inflexion at $x = -2$ minimum at $x = 1$
9. 2 rows of 8
10. base = 3·652 ft. : height = 1·826 ft.
11. 2·1 : minimum
12. $\dfrac{l - y}{2}$
13. ·4968 : £632
14. width = height = 8·4′
15. 15·2 knots : £956, £948, £957
16. $x = \cdot 289l$
17. $h = 6\cdot 34$ ft. : $d = 12\cdot 68$ ft.
18. base = 4″ : height = 5″
19. depth = 3 × breadth.
20. $\dfrac{ry}{l}$
21. 6
22. ·866 r
23. $v : \dfrac{v}{3} : \dfrac{8}{27}$
24. $u = \cdot 5v$
25. 135° or 315°

26. $d = l$ 27. $l = 2{\cdot}065r$ 28. $20{\cdot}15 : -{\cdot}45$ 29. $20^\circ\ 56'$

30. height $= 8{\cdot}1$ ft. : base $= 6{\cdot}72$ ft. 31. $x = {\cdot}4l$

32. $\tan^{-1}\left(\dfrac{-2\mu + \sqrt{2(\mu^2 + 1)}}{1 - \mu^2}\right)$ 33. $l = \sqrt{\dfrac{{\cdot}6f}{K\rho n^2}}$

34. maximum at $x = 0$: points of inflexion at $x = \pm \dfrac{h}{\sqrt{2}}$

35. $\sqrt{P_1P_2}$ 36. $\sqrt{\dfrac{d}{4}(4l + d)}$ 37. $1 + \sqrt{\dfrac{K}{1 + K}}$

38. $\cdot58$ $\left(\text{Let } r = \dfrac{P_2}{P_1} \text{ and find } \dfrac{dW^2}{dr}\right)$ 39. 3 units 40. $x = \dfrac{l - 2r}{2}$

41. $T_f = \frac{1}{3}T_m$ 42. maximum at $x = -2$
minimum at $x = 4$ and at $x = -2{\cdot}5$
points of inflexion when $x = -2{\cdot}26$ and $1{\cdot}92$

43. $x = \sqrt{R_1R_2} : p_x$ (maximum) $= \dfrac{w\omega^2}{8gm}(3m + 1)(R_1 - R_2)^2$

44. $d = \sqrt[4]{\dfrac{D^5}{8fl}}$ 46. $83^\circ\ 1'$ or $276^\circ\ 59'$

Exercises 11, p. 113

1. $\cdot006$ 2. $2{\cdot}5\%$ too low 3. $\cdot0264$ 4. $-2{\cdot}45$

5. $2{\cdot}66$ 6. decrease of $\cdot00135$ 7. $\cdot03$ link; $\cdot237$ link

8. $1 + x \log a + \dfrac{(x \log a)^2}{1 . 2} + \dfrac{(x \log a)^3}{1 . 2 . 3} + \ldots$

9. $2214{\cdot}2525$ 10. $\cdot5344$ 13. $1{\cdot}0358$

11. $3{\cdot}4936$ 12. $7{\cdot}8762$ 14. $1{\cdot}3481$

Exercises 12, p. 131

2. 152 5. 240; 205; 64

6. 621,000 ft. lbs. : potential energy $= 240{,}000$:
kinetic energy $= 381{,}000$: 987,000 ft. lbs.

7. 480 9. 238,000 10. 3006 12. $1{\cdot}526x^{2{\cdot}62} + C$

13. $70{\cdot}15x + C$ 14. $-\dfrac{1{\cdot}5}{x^2} + C$ 15. $\dfrac{x^{2-n}}{2-n} + C$ 16. $1{\cdot}43e^{\cdot7x} + C$

17. $\cdot1x^{10} - 10 \log x + 14x + C$ 18. $3{\cdot}32x^{1{\cdot}04} - 2{\cdot}5x^2 + C$

19. $1{\cdot}074x^{3{\cdot}718} + \frac{1}{4}e^x + C$ 20. $1{\cdot}33e^{9x-6} + C$

21. $6{\cdot}54e^{2{\cdot}4x-1{\cdot}2} + C$ 22. $\cdot689x + C$ 23. $\dfrac{\cdot71bx^{1{\cdot}41}}{c} + C$

24. $3{\cdot}025x^{\cdot84} - 8{\cdot}2 \log x - 2{\cdot}71e^{-2{\cdot}6x} + 1{\cdot}13x + C$

25. $-\cdot0234e^{-10{\cdot}2x} + C$ 26. $1{\cdot}96e^{\cdot51x} - 1{\cdot}297x^{\cdot77} + \cdot674x^{3{\cdot}04} + C$

27. $\cdot797 \cos \theta x^{1{\cdot}18} - 2{\cdot}2e^{8x-14} + C$ 28. $\frac{1}{6}v^6 + C$ 29. $-\dfrac{1}{3u^3} + C$

30. $35t + C$ 31. $\frac{1}{5}e^{5t-4} + C$ 32. $-5{\cdot}88pv + C$

33. $20{\cdot}2(2)^t + C$ 34. $-\dfrac{4}{x} + 2{\cdot}5x^2 + 4{\cdot}25x^4 - 8x + C$

35. $\cdot885(3{\cdot}1)^t + C$ 36. $-\dfrac{5{\cdot}67}{e^{3v}} + C$

37. $\frac{e^{4t}}{4} - \frac{e^{-5t}}{5} - 2{\cdot}718t + C$ **38.** $16{\cdot}1t^2 + C$ **39.** $\frac{\cdot175}{x^2} + C$

40. Write the equation in the form $\frac{dp}{p} = -n\frac{dv}{v}$ and then integrate: $pv^n = C$

Exercises 13, p. 137

1. $-\frac{3}{4}\cos 4x + C$ **2.** $1{\cdot}73 \sin(3 - 3x) + C$

3. $-49 \tan(3 - \frac{1}{7}x) + C$ **4.** $1{\cdot}01x^{\cdot988} + 1{\cdot}195 \sin(\cdot05 - \cdot117x) + C$

5. $\cdot1854e^{5\cdot4x} - \frac{5}{a}\cos(b + ax) + C$ **6.** $9{\cdot}45x \,.\, \sin 8t + C$

7. $\cdot713 \cos 2(2{\cdot}16x - 4{\cdot}5) + C$

8. $12{\cdot}85e^{\cdot7x} - \frac{\cdot868}{x^4} + 1{\cdot}83 \log \cos x + C$

9. $-9{\cdot}95 \cos\left(\frac{3x - 2{\cdot}8}{7}\right) + \cdot022 \sin 9x - 1{\cdot}46x^{2\cdot74} + \cdot455(3)^{2x+8} + C$

10. $2x - \cdot787 \cos\left(\frac{\pi}{4} - 3{\cdot}7x\right) + 7{\cdot}55 \cot\frac{3\pi x}{5} + C$

11. $v = 7 \cos(7t - \cdot26) + C$; $s = -\frac{2}{49}$

12. $\frac{dx}{d\theta} = 4\pi^2n^2r\left(\sin\theta + \frac{\sin 2\theta}{2m}\right) + C$; $x = -4\pi^2n^2r\left(\cos\theta + \frac{\cos 2\theta}{4m}\right) + C$

13. $\cdot311(5^{2p}) - \cdot139 \sin(3{\cdot}7 - 7{\cdot}2p) + C$

14. $-19{\cdot}5 \cos 6t - 4{\cdot}9 \sin 6t + C$

Exercises 14, p. 143

1. $\cdot1825$ **2.** $\cdot345$ **3.** $1{\cdot}7$ **4.** $\cdot5585$ **5.** $\cdot0626$

6. $1{\cdot}218 \times 10^7$ **7.** $2{\cdot}62$ **8.** $\cdot1589$ **9.** $\frac{2I}{p}$

10. $\cdot616B_{max.}$ **11.** $-\frac{1}{2}\left\{\frac{\cos(a+b)t}{a+b} + \frac{\cos(a-b)t}{a-b}\right\}$; 0

12. $\frac{v_1r_1^2}{2g}\left\{\frac{1}{R_1^2} - \frac{1}{R_2^2}\right\}$ **13.** $P = \frac{wl}{8}$; $y = \frac{w}{8EI}\left\{\frac{lx^3}{2} - \frac{x^4}{3} - \frac{l^3x}{6}\right\}$

14. $\frac{w}{24EI}\{x^4 - 4l^3x + 3l^4\}$ **15.** $\frac{w}{24EI}\left\{\frac{l^2x^2}{2} - x^4 - \frac{l^4}{16}\right\}$

16. $\frac{l^5}{30}$ **17.** $\cdot2046l^4$ **18.** $\frac{1 - \sin\phi}{1 + \sin\phi} \cdot \frac{wh^2}{2}$

19. $26{\cdot}24$ {Limits must first be found}; $\frac{3}{4}$ **20.** 334

21. $y = \cdot736x^{3\cdot4} + 5 \log x + 3x - 3{\cdot}25$ **22.** $1{\cdot}087$.

23. $\frac{N}{al}(e^{al} - 1)$ **24.** $\frac{2\pi wV_1^2(R_1^3 - R_2^3)}{3R_1^2}$ **25.** $-\frac{l^3}{8}$

26. $H = \frac{wl^2}{96r}$ **27.** 240 **28.** $2{\cdot}906$ **29.** $\frac{8{\cdot}92}{l^2}\sqrt{\frac{EI}{w}}$

30. $49{\cdot}82$ **31.** $\frac{\pi d t \rho s^2}{12l\mu}$ **32.** $\cdot8596CA_1lh$ **33.** $\frac{\pi f a^4}{8\mu}$

34. $\frac{l}{5}\left[\log\left(\frac{x_1}{x}\right)^2 + \frac{\pi y}{x}\right]$ **35.** $283{\cdot}3$ sq. units **36.** $1{\cdot}487$

37. $635{\cdot}8$ **38.** $1{\cdot}217$ secs. **39.** $\frac{1}{5\pi}$

Exercises 15, p. 178

1. $\cdot 158 \log \dfrac{C(x-1\cdot 583)}{(x+1\cdot 583)}$ 2. $\dfrac{1}{3\sqrt{6}} \tan^{-1} \dfrac{x+1}{\sqrt{6}} + C$

3. $\log (9x^2 - 18x + 17)^{\frac{1}{18}} + C$ 4. $\cdot 1919$

5. 605 {Let $u = 6 - h$} 6. $y\sqrt{\cdot 75 - y^2} + \cdot 75 \sin^{-1} 1\cdot 154y + C$

7. $\cdot 106$ 8. $\frac{1}{4} \sin 12t + 4t + C$ 9. $-\frac{1}{5} \cot 5x + C$

10. $\dfrac{WR^3}{EI}\left(\dfrac{1}{\pi} - \dfrac{1}{4}\right)$ or $\dfrac{\cdot 0683\, WR^3}{EI}$

11. $-\cdot 8 \log \cos 5t + C$ {Let $u = \cos 5t$} 12. $x \sin^{-1} x + \sqrt{1 - x^2} + C$

13. $-\frac{1}{3}\sqrt{(a^2 - x^2)^3} + C$ 14. $\dfrac{5e^{3x}}{13}\{3 \sin 2x - 2 \cos 2x\}$

15. $\tan^{-1} x + \dfrac{x}{1 + x^2} + C$ (Let $u = \tan^{-1} x$) 16. $\cdot 0795$

17. $\frac{1}{3} \sin^3 \theta$ 18. $\dfrac{\cdot 0253 fQ^2}{k}\left[\dfrac{1}{d^4} - \dfrac{1}{(d+kl)^4}\right]$

19. 183 secs.: {Rationalise denominator of right hand side by multiplying top and bottom by $\sqrt{h+12} - \sqrt{h}$; then integrate, making the substitution $u = h + 12$.}

20. $\dfrac{2}{\sqrt{\pi}}\left(t - \dfrac{t^3}{3} + \dfrac{t^5}{10} - \dfrac{t^7}{42} + \ldots\ldots\right)$ 21. $\dfrac{4F}{3\pi r^2}$. (Put $u = r^2 - y^2$)

22. $\dfrac{t^3}{8} + \dfrac{t\sqrt{t^2 - 4}}{8} - \dfrac{1}{2} \cosh^{-1} \dfrac{t}{2} + C$ 23. $\dfrac{35\pi}{256}$ 24. $1\cdot 749$

25. $\dfrac{16}{315}$ 26. $\cdot 01 R^4$

27. $(\cdot 5x + 1\cdot 25)\sqrt{21 - 5x - x^2} + 13\cdot 63 \sin^{-1}\left(\dfrac{x + 2\cdot 5}{5\cdot 22}\right) + C$

28. $F = 2\pi k\sigma\left(\dfrac{z}{\sqrt{a^2 + z^2}} - 1\right)$ 29. $t = \dfrac{L}{2gcz}\left\{\log \dfrac{(z+V_2)(z-V_1)}{(z-V_2)(z+V_1)}\right\}$

Exercises 16, p. 187

1. 41·59 2. 1·718 3. 5·25 4. ·688 5. 4·5
6. 23·05 7. 1·348 ft. candles. 8. 0 9. ·1294
10. 273 11. 205 lbs.

Exercises 17, p. 193

1. 14·14 2. ·0215 3. 2·4 4. ·1165 5. ·825 6. 1·194

7. R.M.S. = ·509 ; $\dfrac{\text{R.M.S.}}{\text{M.V.}} = 1\cdot 11$ 8. $\dfrac{3\cdot 32\sqrt{E_1^2 + E_3^2}}{3E_1 + E_3}$

9. $\dfrac{\text{Sinusoidal}}{\text{triangular}} = \cdot 816$ 10. 5·01 11. 9·86″ 12. ·108 13. ·14

Exercises 18, p. 209

1. 1450 cu. yds. 2. 5977 lbs. 3. 1071 4. $4\pi r^2$

5. $a = 3{\cdot}036$, $b = {\cdot}1423$; 1617 6. $\dfrac{K\pi D^2 l^2}{8}$ or Vol. $\times \dfrac{Kl}{2}$

7. 524 {limits are 5 and 10} 8. 271·6 9. 1·2 lbs.

10. true = 76·62 : (a) 75·41 : 1·58 % low (b) 77·73 : 1·45 % high (c) 76·60 : correct 11. 60·9 ft. 12. 1011 lbs.

Exercises 19, p. 234

1. $\frac{5}{6}$ height from vertex
2. area = 7200 sq. ft.: centroid is 158′ from forward end
3. 3·84 lbs.; 4″ 4. 10 ft. from top
5. $\bar{x} = \bar{y} = 1{\cdot}7''$ (taking the centre as origin) 6. 771; 2·25
7. (0, ·95) 8. 2·35″ from AB 9. 1·055″ 10. ·935″
11. ·877″ 12. 1·02″ 13. 2·68″ *i.e.*, $1{\cdot}34l$
14. 5·12 ft. from O 15. 30880 lbs. 16. 18432 lbs.: 3·534 ft.
17. 9 units. 18. 5·1 ins. 19. 2·35 ins.
20. (a) 17 lbs. (b) $1\frac{3}{4}''$ 21. 1·48 ins. 22. $\frac{2}{5}h$ 23. $\frac{3}{4}h$

Exercises 20, p. 254

1. $\cdot 655l$ 2. 17·14 in.: 25420 lbs. in.^{2}
3. C. of G. is ·1125″ distant from centre of large circle: 2·68″
4. 16·6 ins.^{4} 5. (a) $\cdot 408h$; (b) $\cdot 707h$
6. $I_{AB} = 80{\cdot}7$ ins.^{4}: $k_{AB} = 3{\cdot}04''$ 7. 29·1″ 8. 377
9. 9·86 10. 7·35
11. $\cdot 2887d$. (Divide into strips by planes perpendicular to the axis and sum the polar moments of these)
12. $\dfrac{\text{IE of circular}}{\text{IE of square}} = \dfrac{3}{\pi} = {\cdot}956$ 13. 33·3 inch units
14. 11613: 17·04
15. $I_{NN} = 169{\cdot}4$: $k_{NN} = 2{\cdot}44$: $I_{AB} = 570$: $k_{AB} = 4{\cdot}47$
16. NN is 3·99″ from bottom of lower flange: $I_{NN} = 461$: $k_{NN} = 3{\cdot}77$
17. 5·04″ 18. 2·023′: ·444 19. 2·74″
20. (a) 13·9 (b) 31·1 (c) 1·48″ (d) 1·82″ 21. 1·60
22. mass$\left(R^2 + \dfrac{3r^2}{4}\right)$ 23. $\cdot 5b$ 24. ·2449 F.P.S. unit

Exercises 21, p. 268

3. 5·23 4. $p = 2r \sin\theta$: the sine curve 6. 892 7. 5·01

Exercises 22, p. 295

1. $y = 1{\cdot}67x^2 - 2{\cdot}4x - 12{\cdot}82$ 2. $s = 8{\cdot}05t^2 - 23{\cdot}07t + 14{\cdot}09$
3. $y = Ae^{3x} - \frac{1}{3}$ 4. $y = 8{\cdot}35 - 6{\cdot}18e^{-1{\cdot}044x}$
5. $y = \dfrac{W}{48IE}(6lx^2 - 4x^3 - l^3)$ $\left(\dfrac{dy}{dx} = 0 \text{ when } x = 0;\ y = 0 \text{ when } x = \dfrac{l}{2}\right)$

6. $K = \frac{wl^2}{12}$: $y = \frac{w}{24EI}\left(\frac{l^2x^2}{2} - x^4 - \frac{l^4}{16}\right)$

7. $\log\left(\frac{T_2 - t}{T_1 - t}\right) = \frac{\pi r l D}{as(1 + r)}$ 8. $v = \frac{p}{8l\mu}(s^2 - x^2)$

9. $\log\left(\frac{\tau_1 - \theta}{\tau_2 - \theta}\right) = \frac{\pi QDl}{ws}$ 10. $\frac{\pi QDl}{ws} = \frac{1}{\tau_2} - \frac{1}{\tau_1}$

11. $H = \frac{2\pi Kl(\tau_1 - \tau_2)}{\log \frac{r_2}{r_1}}$ 12. $\theta = A\sin\left(\sqrt{\frac{mh}{I}}t + c\right)$

13. $p = A + \frac{B}{r^2}$ 14. $p = \frac{b}{x^3} - a$ 16. $\log\frac{z}{C} + \frac{w\omega^2x^2}{2gf} = 0$

17. $y = A_1e^{10x} + A_2e^{7x}$: $y = A_1e^{10x} + A_2e^{7x} + 1$

18. $s = A_1e^{9\cdot 33t} + A_2e^{-9\cdot 33t}$ 19. $s = A\sin(9\cdot 33t + B)$

20. $KR\log\frac{v_1}{v_2}$ 21. $C = C_0e^{-\frac{Rt}{L}}$ 22. $C = \frac{V_0\sin\left(qt - \tan^{-1}\frac{Lq}{R}\right)}{\sqrt{R^2 + L^2q^2}}$

23. $x = a\log A\left(\frac{y + a + \sqrt{y^2 + 2ay}}{a}\right)$ or $\frac{a}{A}e^{\frac{x}{a}} = y + a + \sqrt{y^2 + 2ay}$

(Separate the variables and use the $\int\frac{dx}{x^2 - a^2}$ form)

24. $x = A\sin\left(\sqrt{\frac{a}{SM^2}}t + c\right) + pS$ 25. $y = \frac{Y\sin\sqrt{\frac{P}{IE}}x}{\sin\sqrt{\frac{P}{IE}}\frac{L}{2}}$

26. $nt = \log A\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)$: $\cdot 1945$ 27. $x = \cdot 833e^{-7\cdot 485t}\sin 12t$

28. $x = Ae^{-3t}\sin(6\cdot 32t + c) + \cdot 026a\sin(5t - \tan^{-1}1\cdot 25)$

29. $V = A_1e^{\sqrt{\frac{r_1}{r_2}}x} + A_2e^{-\sqrt{\frac{r_1}{r_2}}x}$ 30. $pv^n = C$

31. $\theta = \theta_a + e^{-kt}(\theta_0 - \theta_a)$ 32. $\theta = -53\cdot 65e^{-1\cdot 16t} + 5\cdot 83e^{-1\cdot 065t} + 48\cdot 77$

33. $y = A_1e^{jax} + A_2e^{-jax} + A_3e^{jbx} + A_4e^{-jbx} - \frac{g}{\omega^2}$

where $a = \sqrt{\frac{F + \sqrt{F^2 + \frac{4\rho}{g}\omega^2EI}}{2EI}}$

and $b = \sqrt{\frac{F - \sqrt{F^2 + \frac{4\rho}{g}\omega^2EI}}{2EI}}$

or $y = B_1e^{cx} + B_2e^{-cx} + B_3\sin\phi x + B_4\cos\phi x - \frac{g}{\omega^2}$

34. $v = \frac{g\cos\alpha}{k}(1 - e^{-kt})$ 35. $t = \frac{2\cdot 5}{a}\log\frac{C(5 + v)}{(5 - v)}$

36. $y = 2e^{10x} + 3e^{10x}$ 37. $y = (c + x)e^{5x}$
$y = (c - e^{-x})e^{5x}$

Exercises 23, p. 339

1. 103 secs. if coefficient of discharge is taken as ·62 **2.** 7·81
3. 2·83 cu. ft. **4.** 69·5 : 95·5 (Draw in the "simply supported" bending moment diagram and work on the Goodman plan, see page 313)
5. Find the time to lower to level of upper orifice (183 secs.) with both orifices open; then the time for the further lowering of 5 ft., through the one orifice (180 secs.). Total time = 363 secs. Note that $\dfrac{1}{\sqrt{h+12}+\sqrt{h}} = \dfrac{\sqrt{h+12}-\sqrt{h}}{12}$ **6.** 57·7 secs. **7.** 1·4
8. 5500 lbs.: 4·71 ft. below S.W.S.L. **9.** 14100 lbs.: 6·65′ below S.W.S.L. **10.** 723·5 **11.** $\dfrac{\cdot 0253 f Q^2}{K}\left(\dfrac{1}{d_e^4} - \dfrac{1}{(d_e+Kl)^4}\right)$ (Hint, let $u = d_e + Kx$) **12.** 1·23 **13.** Vertical depth = 8·07 ft.

Exercises 24, p. 351

1. $x = 2{\cdot}31 - 1{\cdot}231 \cos\theta - 1{\cdot}55 \sin\theta - {\cdot}16 \cos 2\theta$
$- {\cdot}022 \sin 2\theta - {\cdot}004 \cos 3\theta - {\cdot}04 \sin 3\theta$
2. $A = 1{\cdot}29$, $a_1 = 0$, $B = {\cdot}14$, $a_2 = \pi$
3. $y = 16{\cdot}52 - {\cdot}18 \sin x + 5{\cdot}46 \cos x - 2{\cdot}02 \sin 2x$
$- 13{\cdot}71 \cos 2x - 1{\cdot}12 \sin 3x - 3{\cdot}25 \cos 3x - {\cdot}43 \sin 4x - 1{\cdot}21 \cos 4x$
4. $E = 1500 \sin\theta + 100 \sin 3\theta + 42 \cos\theta + 28 \cos 3\theta$

Exercises 25, p. 367

1. B = 39° 31′; A = 65° 51′; c = 57° 5′
2. c = 54° 44½′; b = 34° 14′; B = 43° 32½′
3. A = 161° 8′; B = 13° 35′; C = 9° 38′
4. A = 76° 36′; B = 64° 8′; c = 52°
5. B = 35° 43′ 40″; A = 61° 21′ 20″; a = 43° 25′ 23″
6. C = 33° 29′ or 146° 31′
a = 103° 28′ or 55° 28′
A = 146° 58′ or 27° 30′ **7.** 34° 52′ **8.** 342,200
9. Latitude = −2° 12′. hour angle = 4° 31·3′

Exercises 26, p. 388

1. 2nd set better than 1st set in the proportion 1·943 to 1: 6 chns. 43·4 links. **2.** Just under 1. **3.** 98° 6′ 42·26″ : r_m = ·307″, r = 1·062″
4. 96·86 **5.** T = 26·71 + 3518α **6.** ·5479

1. **Determine the indefinite integral** $\int \frac{dx}{x^2 - 12x}$.

$$\left[\log C\left(\frac{x-12}{x}\right)^{\frac{1}{12}}\right]$$

2. The flux per cm. length between two parallel conductors each carrying the same current I is given by $\int_r^{d-r}\left(\frac{2I}{x}+\frac{2I}{d-x}\right)dx$ where r = radius of wires and d is the distance between their centres. Find the value of the flux when $d = 30$, $r = \cdot 6$ and $I = \cdot 25$. [3·892]

3. **Evaluate the definite integral** $\frac{1}{\pi}\int_0^{\pi} 1440 \sin^2\theta \, d\theta$, **giving the power absorbed by a thermionic valve rectifier.** [720]

4. **Solve the differential equation** $\frac{dy}{dx} + \frac{3y}{x} = 4y^2$.

[Hint.—The equation $\frac{dy}{dx} + Py = Qy^n$ (Bernouilli's equation) is converted into the standard form by the substitution $z = y^{-(n-1)}$. In this case $n = 2$ and $z = y^{-1}$ so that $\frac{dy}{dx} = \frac{dz^{-1}}{dz} \cdot \frac{dz}{dx} = -\frac{1}{z^2} \cdot \frac{dz}{dx}$, and the given equation becomes $-\frac{1}{z^2}\frac{dz}{dx} + \frac{3}{zx} = \frac{4}{z^2}$, which must be solved for z]

$$\left[y = \frac{1}{2x + Cx^3}\right]$$

5. Solve the differential equation $\frac{dy}{dx} + \frac{y}{2x} = \frac{8e^x}{y}$.

$$\left[y^2 = 16e^x - \frac{16e^x}{x} + \frac{C}{x}\right]$$

6. **A displacement s is given in terms of the time t by the equation**

$$s = 10 \sin 8t + 4 \cos 8t.$$

Find a value of t for which the velocity is zero and find also the acceleration when $t = \cdot 05$. [·1488 : — 484·9]

7. The magnetic force H on a needle due to a current in a coil distant r from the needle is given by

$$H = \frac{2\pi\, ni \cos^3\theta}{R} \text{ where } \theta = \tan^{-1}\frac{r}{R}.$$

If $l = 2\pi n R$ express H in terms of l, i, r and θ, and then find a value of θ for which H is maximum. Give also the maximum value of H.

$$\left[54^\circ 44'; \ \frac{2li}{3\sqrt{3}r^2}\right]$$

8. If $i = I \sin \omega t + Ae^{-\frac{t}{RK}}$ find the value of $RK\frac{di}{dt} + i$.

$$[I\sqrt{1 + R^2K^2\omega^2}\sin(\omega t + \tan^{-1} RK\omega)]$$

9. Given that $T \sin\theta = W$ and $T\cos\theta = 6(2\cos\theta - 1)$ find a value of θ that gives a maximum value to W. [37° 27′]

10. If $\dfrac{dI}{dl} = EY$ and $\dfrac{dE}{dl} = IZ$, solve for I, the value of the current at distance l from the receiving end of a transmission line.

If the length of the line is L, the current at the sending end is I_s and the voltage at the sending end is E_s, find I_r, the current at the receiving end, in terms of I_s and E_s.

$$\left[I = A_1 e^{l\sqrt{YZ}} + B_1 e^{-l\sqrt{YZ}};\ I_r = A_1 + B_1 = I_s \cosh L\sqrt{YZ} - \sqrt{\frac{Y}{Z}}\,E_s \sinh L\sqrt{YZ}\right]$$

11. Prove that

$$\sinh jx = j \sin x$$
$$\sin jx = j \sinh x$$
$$\cosh jx = \cos x$$
$$\cos jx = \cosh x$$

and thence evaluate $\sinh(\cdot 8 + \cdot 2j)$ [$\cdot 8702 + \cdot 2658j$]

12. Given that $Z = \cdot 6 + \cdot 8j$, $Y = 5 \times 10^{-6} j$, and $L = 100$, find $\cosh L\sqrt{YZ}$. ($\cosh \cdot 0707 \triangleq 1\cdot 0025$, $\sinh \cdot 0707 \triangleq \cdot 07$)

[$\cdot 98 + \cdot 0147j$]

13. If $Ri + L\dfrac{di}{dt} + \dfrac{1}{C}\displaystyle\int i\,dt = E$ where E is constant show that $\dfrac{d^2i}{dt^2} + \dfrac{R}{L}\dfrac{di}{dt} + \dfrac{1}{LC}i = 0$.

Solve this equation for i if $L = \cdot 2$H, $C = 5\mu$F, t is in seconds, for the cases in which (i) $R = 2020\Omega$ (ii) $R = 400\Omega$ (iii) $R = 240\Omega$.

In case (iii) find the time that elapses before the amplitude of the oscillations is reduced to $\frac{1}{10}$th of the initial amplitude.

[(i) $i = Ae^{-10000t} + Be^{-100t}$ (ii) $i = (A + Bt)e^{-1000t}$
(iii) $i = e^{-600t}(A \sin 800t + B \cos 800t)$: $\cdot 00384$ sec.]

14. Find a Fourier series for a function which has the value $\sin x$ from $x = 0$ to $x = \pi$ and $-\sin x$ from $x = \pi$ to $x = 2\pi$, a rectified wave.

$$\left[f(x) = \frac{4}{\pi}\left(\frac{1}{2} - \frac{\cos 2x}{1.3} - \frac{\cos 4x}{3.5} - \frac{\cos 6x}{5.7} - \ldots\right]\right.$$

INDEX